THE INFLUENCE OF TOPOGRAPHY ON STEADY CURRENTS

AND INTERNAL WAVES

by

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## ABSTRACT


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Submitted to the Department of Earth and Planetary Sciences, Massachusetts Institute of Technology and the Woods Hole Oceanographic Institution on January 14, 1971 in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

Observations of the ocean in the vicinity of Bermuda on two different occasions show systematic distortions of the isotherms close to the island and an area of intensive mixing on the northern coast. Two mechanisms are investigated and each produces some agreement with data from different flow regimes.

Firstly, the island is modeled as a circularly symmetric obstacle with steep sides and a small aspect ratio. A steady, rotating, and stratified flow which, far from the island, is uniform in the horizontal and a linear function of the vertical coordinate is taken to be flowing past the island. Neglecting circulation effects, the problem is solved to first order in a small parameter, $\alpha$ , which measures the steepness of the island and a small Rossby number, $\epsilon$ . This allows a calculation of the depth contours of isotherms to $O(\epsilon^2, \epsilon\alpha)$. For one set of data the flow is such that the slope effect of $O(\epsilon\alpha)$ predominates while for another period of observation both slope and Rossby number influences are of the same magnitude. In both cases qualitative agreement between fact and theory is remarkably good. In addition, it is shown that the north slope (for a west-east current) is the most favored area for mixing as there the Richardson number is a minimum and the flow is most likely to separate from the boundary.

A second means of producing isotherm distortion and mixing areas close to the island concerns the nonlinear effects of shoaling internal gravity waves. For normal incidence on a two-dimensional beach the Reynolds stresses produced by the fundamental wave motion are shown to force a mean Eulerian current which is equal but opposite in sense to the Stokes drift. This causes the mean Lagrangian current to vanish so that the physical constraint that there be no net motion of fluid particles along isopycnals into the beach is satisfied. In addition, isotherms are distorted in a fashion analogous to the surface set-down produced by shoaling surface waves. The mean isopycnal shift can be as much as 10m where the theory has some validity. Distortions of the predicted form are observed in the data from a period when the mean currents were small. Consideration of the oblique incidence problem shows that this generalization has little effect on the expected magnitude of the shifts but that a significant longshore current can be forced by the breaking of the waves.

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## 1. INTRODUCTION

In October of 1968 and July of 1969 two cruises, Atlantis II 47 and Gosnold 144 respectively, were undertaken to the Bermuda area with the scientific leadership of Prof. Carl Wunsch of M.I.T.. The purpose of these expeditions was to make detailed oceanographic observations of the oceanic climate in the vicinity of the island in order to determine the island's influence on the surrounding ocean. A more comprehensive description of the data that was gathered and the understanding of it will be presented by Wunsch (1971). This report concentrates on the explanation of a particular feature that was evident in both sets of data.

R. V. Atlantis II was in the vicinity of Bermuda from Oct. 15 to Oct. 29 during which time two instrumented moorings were deployed and 51 S.T.D. stations performed. The positions of these stations, numbered chronologically (1-12 were taken elsewhere and 70-74 were of a different nature), and the moorings, T1 and T3, are shown in figure (1). As can be seen the 51 S.T.D. stations were taken on a series of ten spokes radiating from the island with about 5 stations per spoke and the position on each spoke chosen to be as closely as possible at the odd hundreds of fathoms depth contours (i.e. 100f, 300f, etc.).

Unfortunately, because of the inexperience of the operators, a 10m ambiguity exists in the surface position on the salinity and temperature depth traces obtained from the S.T.D.. However, several interesting features are still evident.

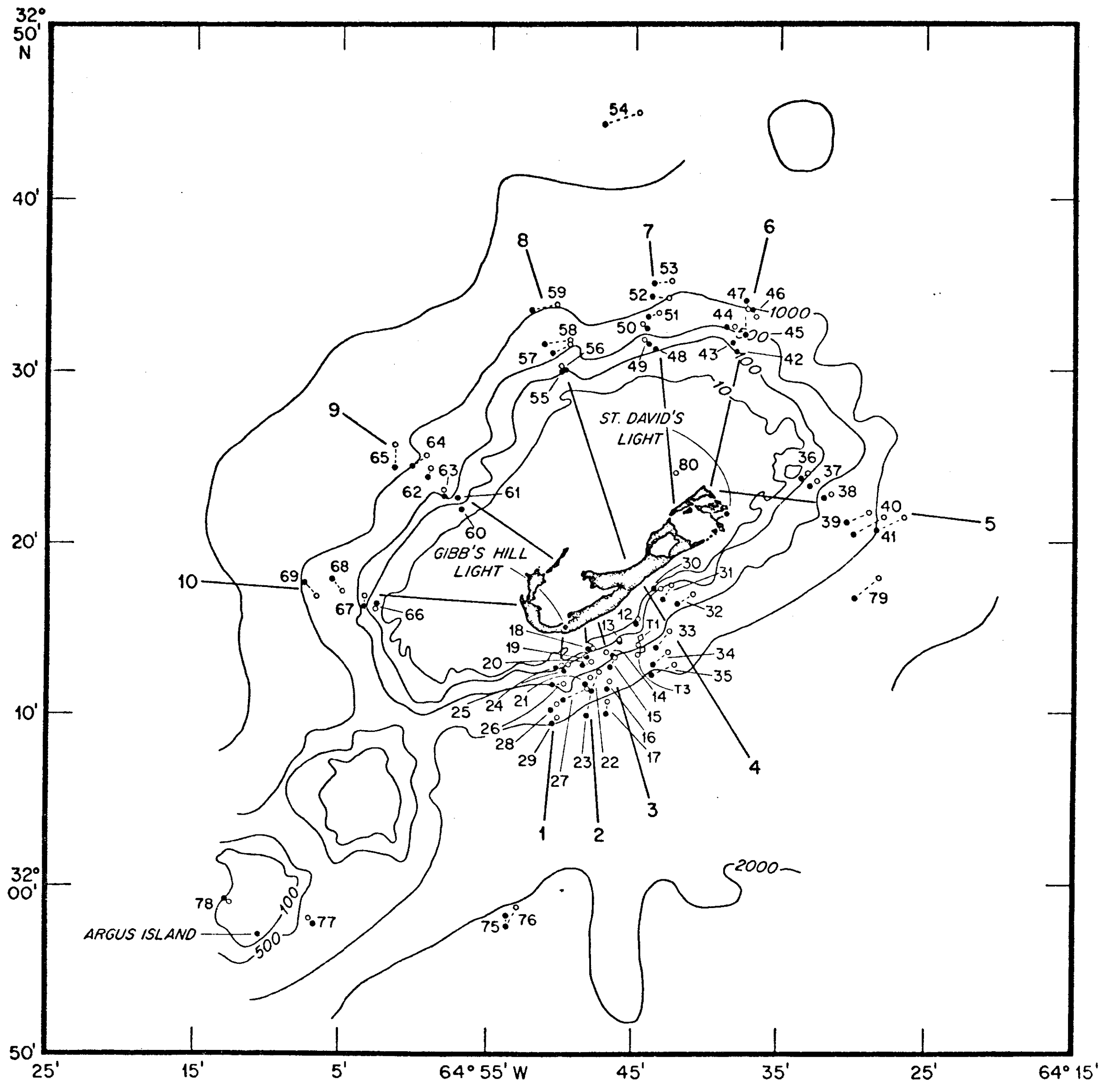



Figure 1. S.T.D. station and current meter mooring positions for the ATLANTIS II-47 cruise.

In particular, there is a marked tendency for isotherm (and isohaline) levels to shift either up or down close to the island by amounts in excess of the 10m ambiguity. For instance, there is a depression of the 18°C isotherm in a small region centered on spoke (8) while the isotherms rise on spoke (4) as the island is approached. Figures (2) and (3) are plots of the depths of the 18°C and 11°C isotherms respectively taken from stations on approximately the same depth contour and plotted against the angle that the spoke makes with the apparent downstream flow direction.

There are several indications of the direction of current flow. The initial and final station positions are given on figure (1) by closed and open circles respectively, and these generally show a surface current towards the ENE at about 20 cm/sec. This is especially true for those stations furthest from the island. In addition, a few current meter stations were taken with a shipborne Marine Advisers current meter. After being corrected for ship's drift the current at all depths is in the same direction as the surface flow with a magnitude that decreases slowly to negligible values at 1km depth. A typical corrected current plot is given in figure (4). Finally, the instrumented mooring T1 had current meters at 610m and 700m and T3 contained one working meter at 690m. This data is somewhat confusing having for part of the time a weak mean flow directed perpendicular to the depth contours but, for the most part, the flow is along the isobaths opposite to the indicated surface current and the shipborne meter data. The magnitude

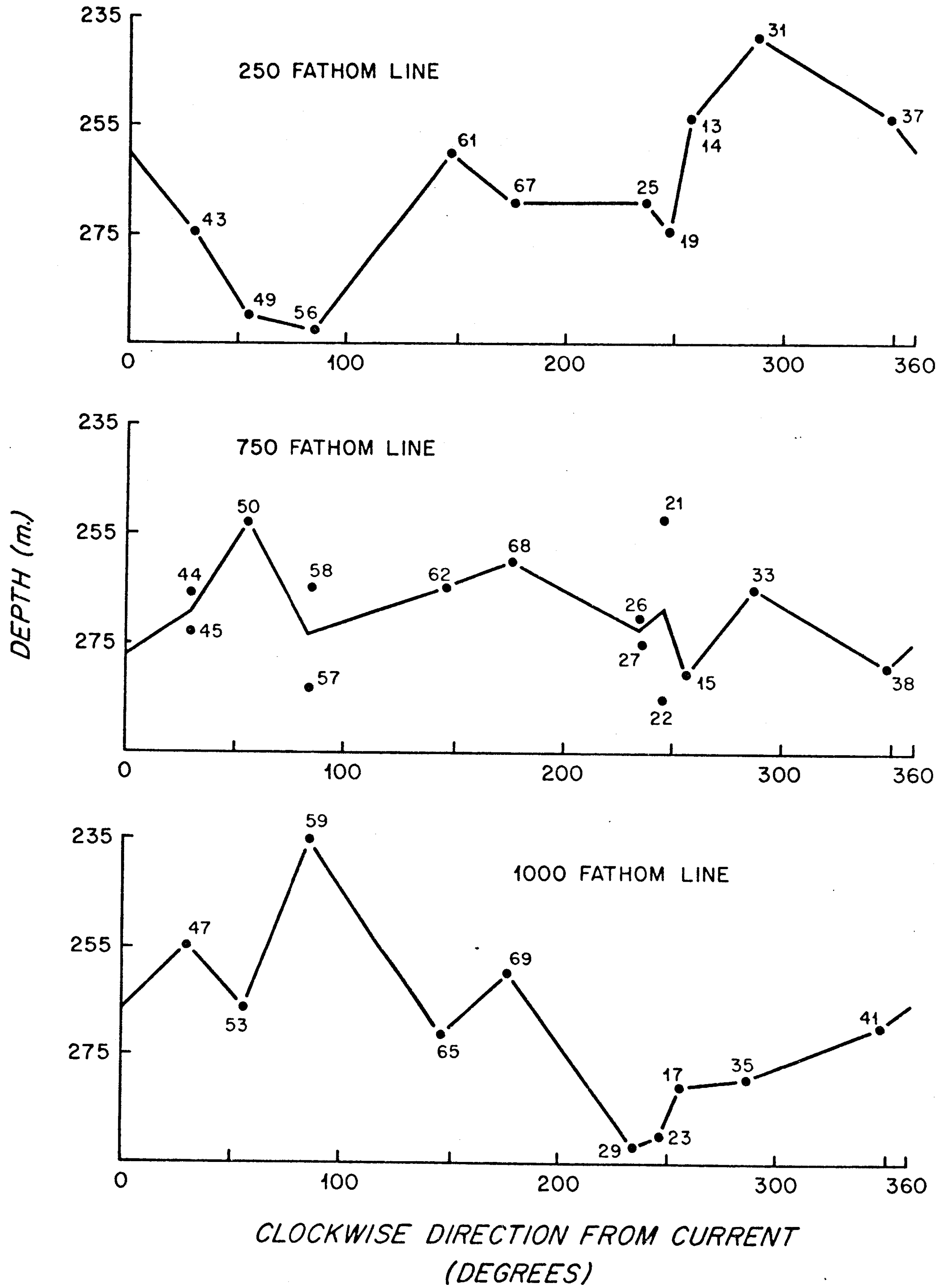



Figure 2. Depth of the 18°C isotherm at selected isobaths around Bermuda for ATLANTIS II-47.

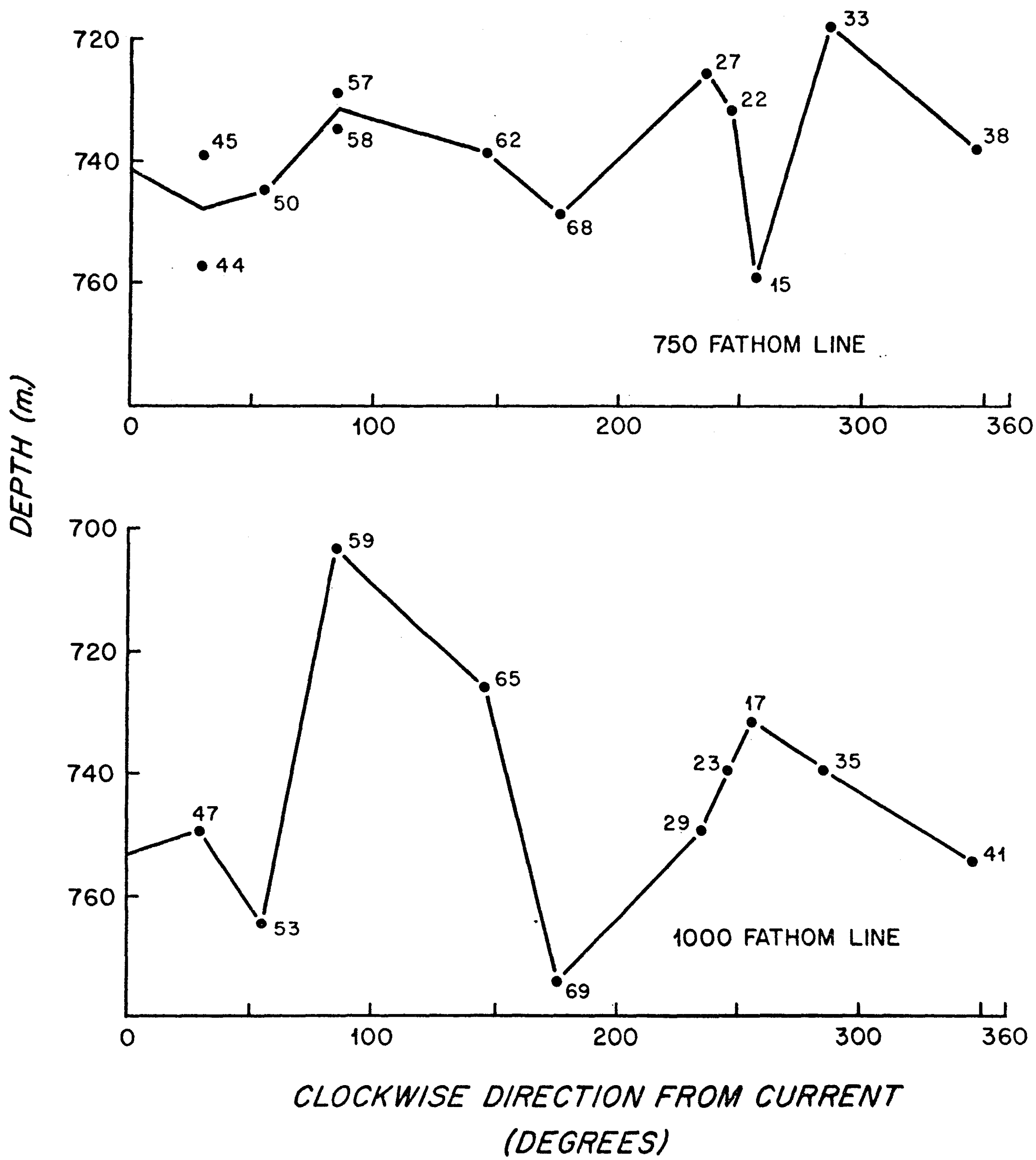



Figure 3. Depth of the 11°C isotherm at selected isobaths around Bermuda for ATLANTIS II-47.

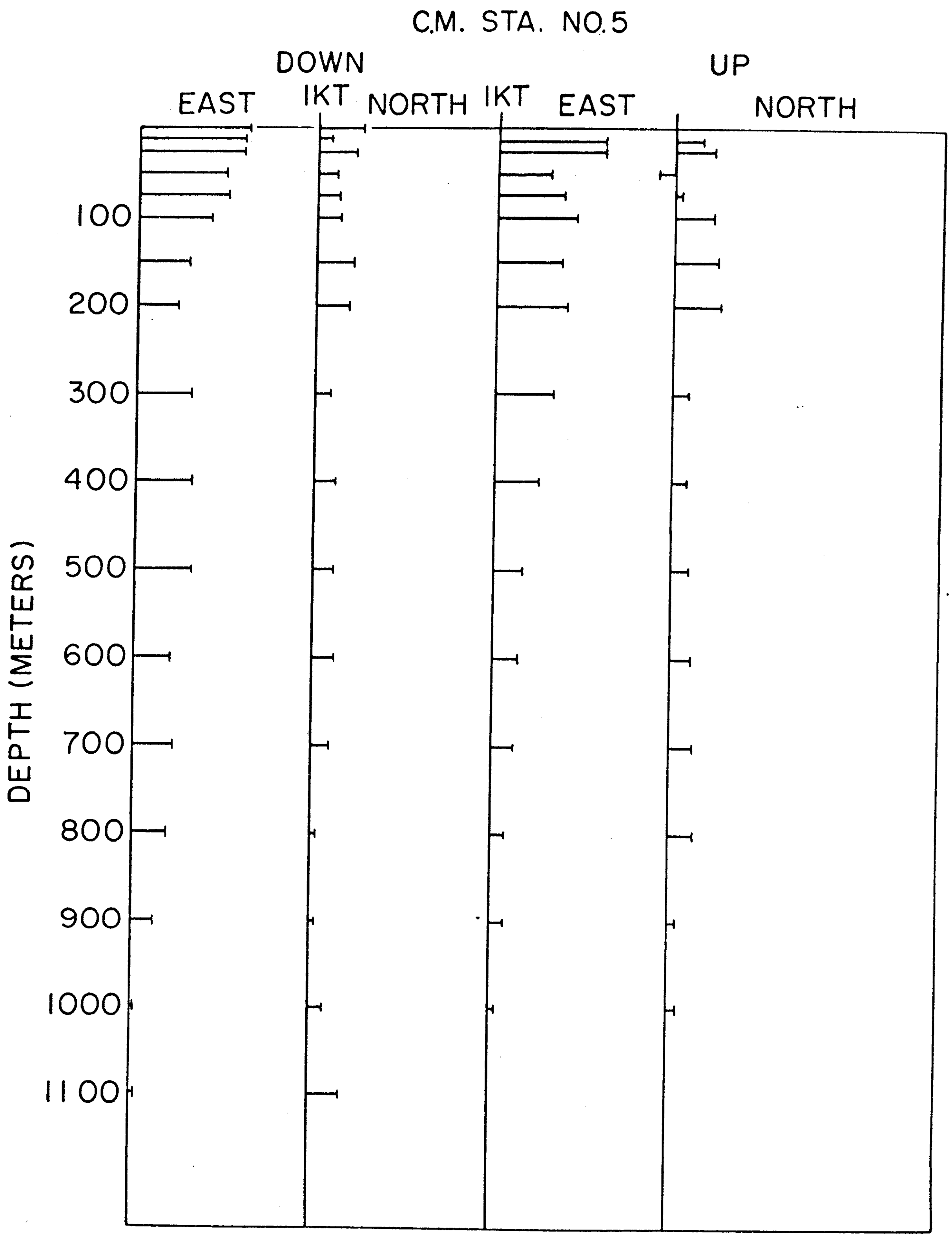



Figure 4. A sample current meter station from ATLANTIS II-47.

is weaker ($<$10cm/sec) probably meaning that the ship drift correction to the ship borne instrument is not wholly correct.

The temperature data for the deepest isobath presented in figures (2) and (3) seems to support the belief that the current is towards the ENE. Isotherms are higher to the north and lower to the south which, if a thermal wind balance holds, indicates that the current shear is in the direction assumed. However, at the shallowest isobaths the reverse isotherm pattern is observed with the deepest isotherm position on Bermuda's northern slope and the shallowest on the southern with differences as large as 60m. The central idea behind this thesis is that the distortion is produced by an interaction of the island with the ocean around it. Two different mechanisms will be studied.

Another, perhaps unrelated, feature was discovered in the S.T.D. data. The stations taken in an area to the north of Bermuda and centered on spoke (7) contained a great deal more microstructure or layering than anywhere else around the island. Figure (5) shows the temperature profiles for the stations that comprise spoke (7). Note the development of thick isothermal layers close to the island. If this phenomenon can be taken as evidence of mixing, then the intensity of the mixing process appears to increase towards shore. Some speculations on the mechanism will be made.

A second expedition, Gosnold 144, was in the area from July 6 to Aug. 1, 1969 and a similar investigation was conducted. This time 63 S.T.D. stations were performed with the majority located in the north slope region where the mixing

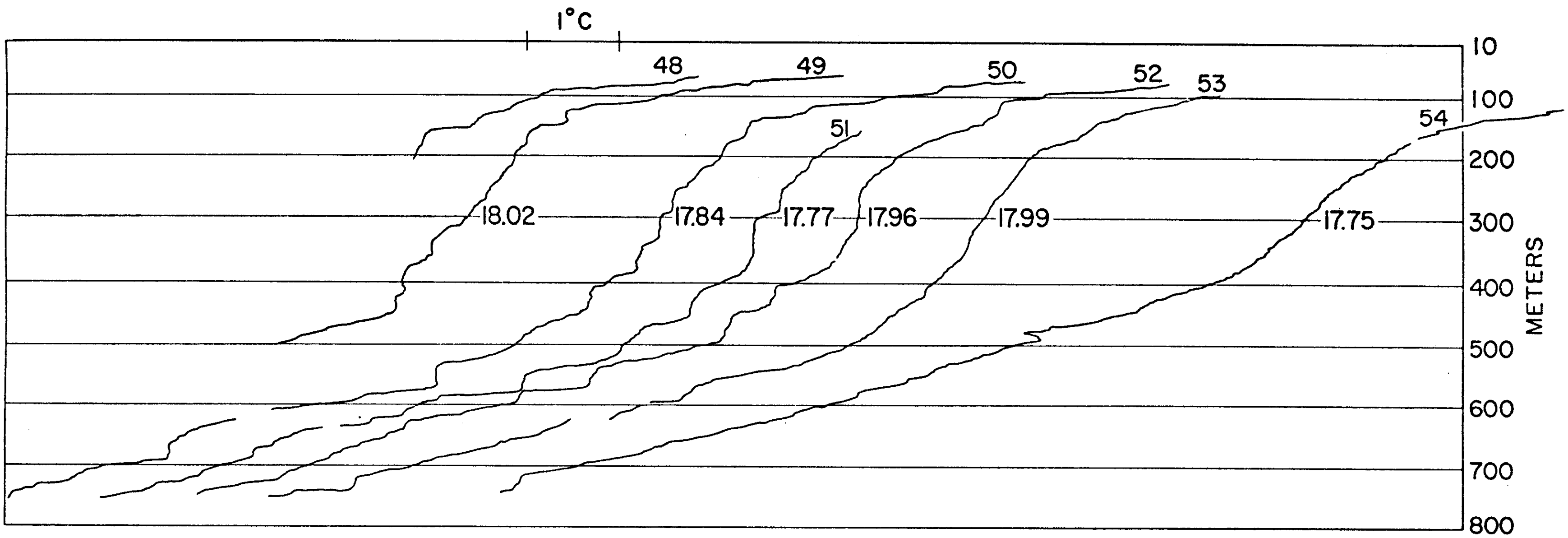



Figure 5. Temperatures profiles from spoke 7 of the ATLANTIS II-47 cruise showing extensive layering.

area had been observed the previous fall. Both this feature and the isotherm distortion near the island were again evident. Figure (6) gives the positions of the STD stations. The peculiar mixing region was located slightly to the west of its position on the previous cruise.

The period that Gosnold 144 was in the vicinity of Bermuda can be divided into two distinct current regimes. When stations 1 to 20 and, perhaps, 21 to 33 were performed the surface flow, as indicated by the ship's drift, was towards the southwest at about 20 cm/sec. For the remainder, conditions had changed substantially and a strong current of over 1k was flowing to the east. As the majority of the stations were performed in this latter period, and most of these were stationed in the interesting regions noted in Atlantis II 47 only their results will be given in any detail. Stations 21 to 33 are included as they are the only stations on the southern coast but it must be remembered that they appear to be part of a transitional period between the two current regimes.

The experience of the STD operators was somewhat improved and the data is accurate to at least ±5m. A larger source of error is inherent in the indicated station positions especially at great distances from the island where dead reckoning was resorted to at times.

An increased confidence in the data made plotting of isotherm depth contours a reasonable approach to the data analysis problem. The station network is not sufficiently dense, however, to allow an unambiguous interpretation of the data.

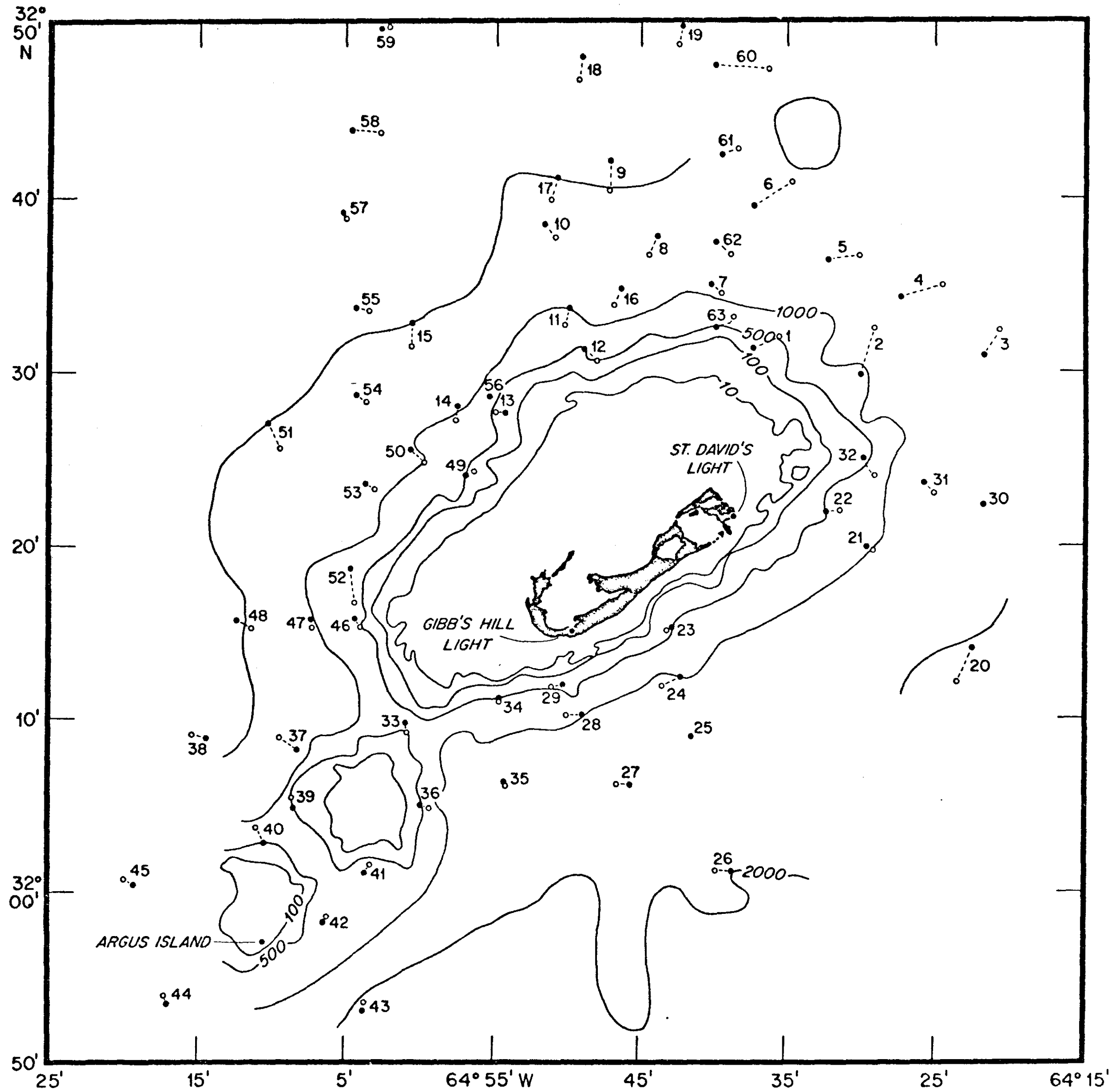



Figure 6. S.T.D. station positions for GOSNOLD-144.

Figures (7) and (8) give two possible methods of contouring the depth data on the 17°C isotherm. The first (figure (7)) is based on a "maximum simplicity" priciple while the second interpretation arises from ideas presented in chapter (2). In either case, there is a rather localized depression of the isotherms on the northwestern slope of Bermuda that can either be treated as a closed "hole" or alternatively joined to deeper isotherm positions far from the island to create a closed "hill" to the south. These same holes and hills show up in the depth contours in figures (9) to (14) for the 14°C, 10°C and 6°C isotherms. The contrast that these topograhic features make decreases with depth.

For indications of the current direction on this cruise, there are the station drift lines shown on figure (6) which connect the initial (closed circle) and final (open circle) station positions. These indicate a predominantly easterly flow for stations 33 to 63. In addition 15 parachute drogues were deployed at nominal 300m and 400m depths. Their tracks are given in figure (15). Drogues (1), (2) and (3) were set while STD stations 1 to 33 were being performed before the strong easterly flow had fully developed. Note the southerly drift of drogues (1) and (3) and the difference between (2) and (13) which were set in almost identical positions. Drogue (4), to the south of the island , was observed over a period of four days and appears to be caught in a shadow zone behind one of the banks. Drogue (5) also appears to be influenced by the banks but the remainder have a velocity of about 1k more or

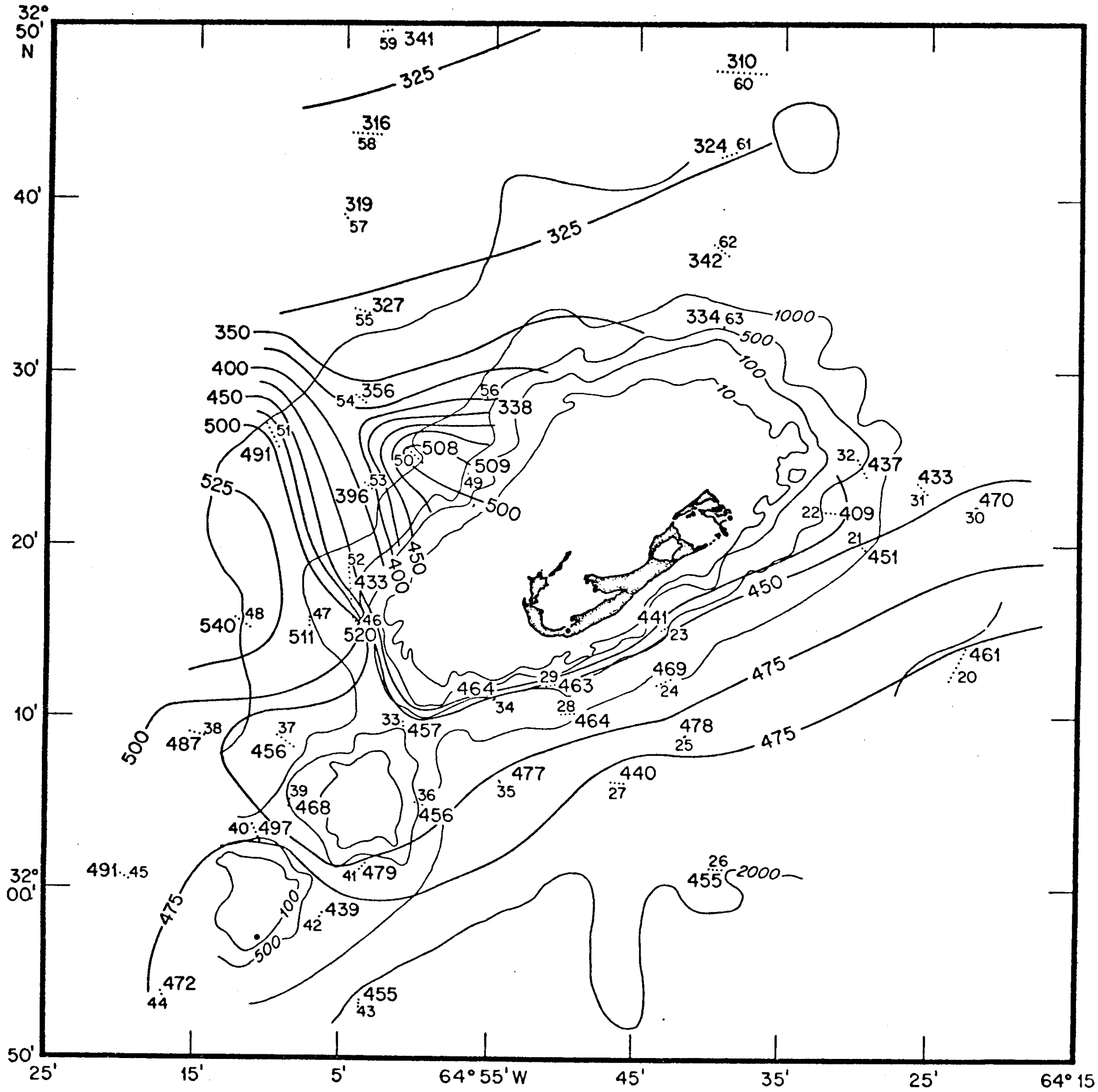



Figure 7. Depth to the 17°C isotherm in meters for GOSNOLD-144.
Interpretation A.

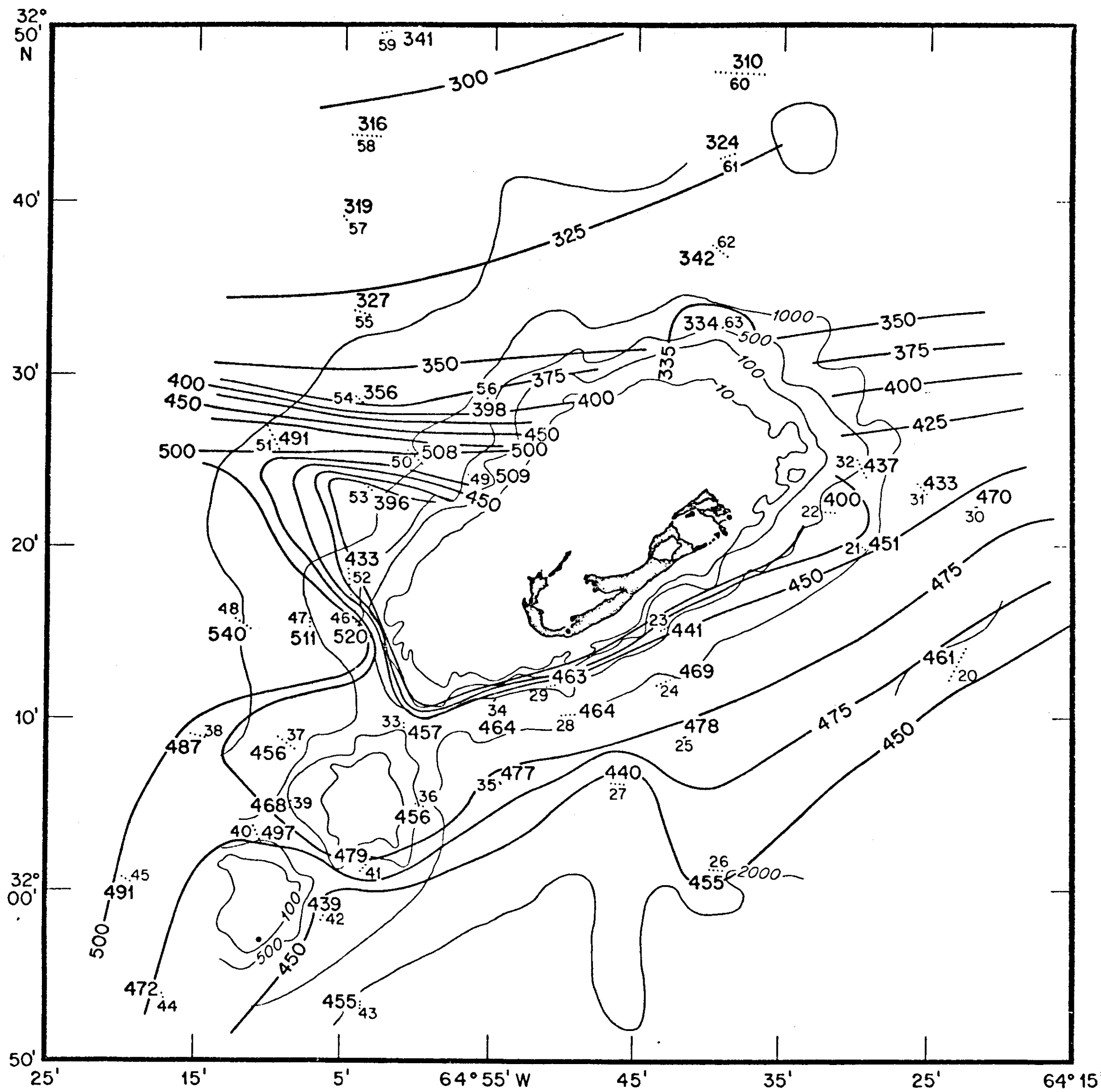



Figure 8. Depth to the 17°C isotherm in meters for GOSNOLD - 144. Interpretation B.

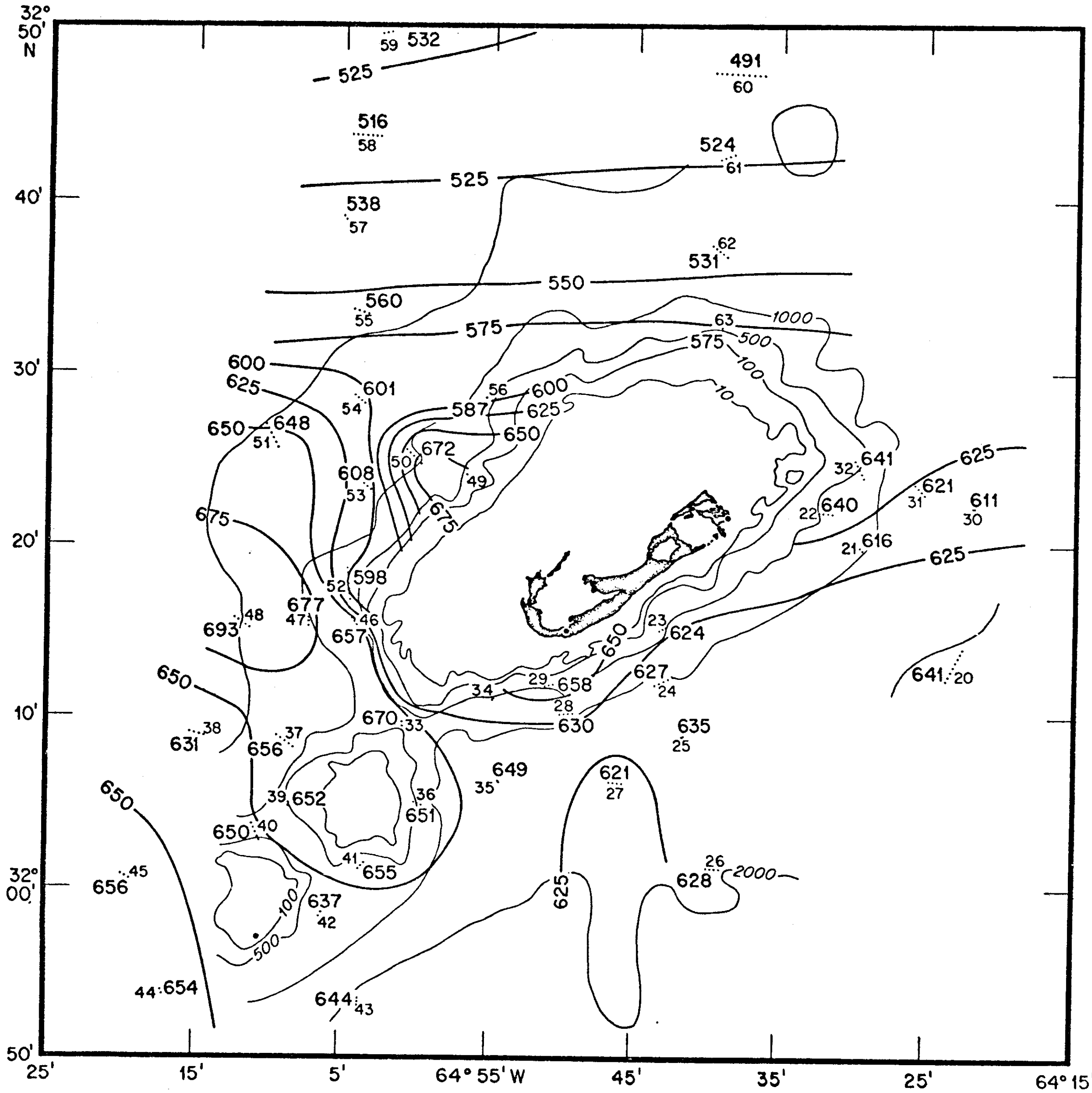


Figure 9. Depth to the 14°C isotherm in meters for GOSNOLD-144. Interpretation A.

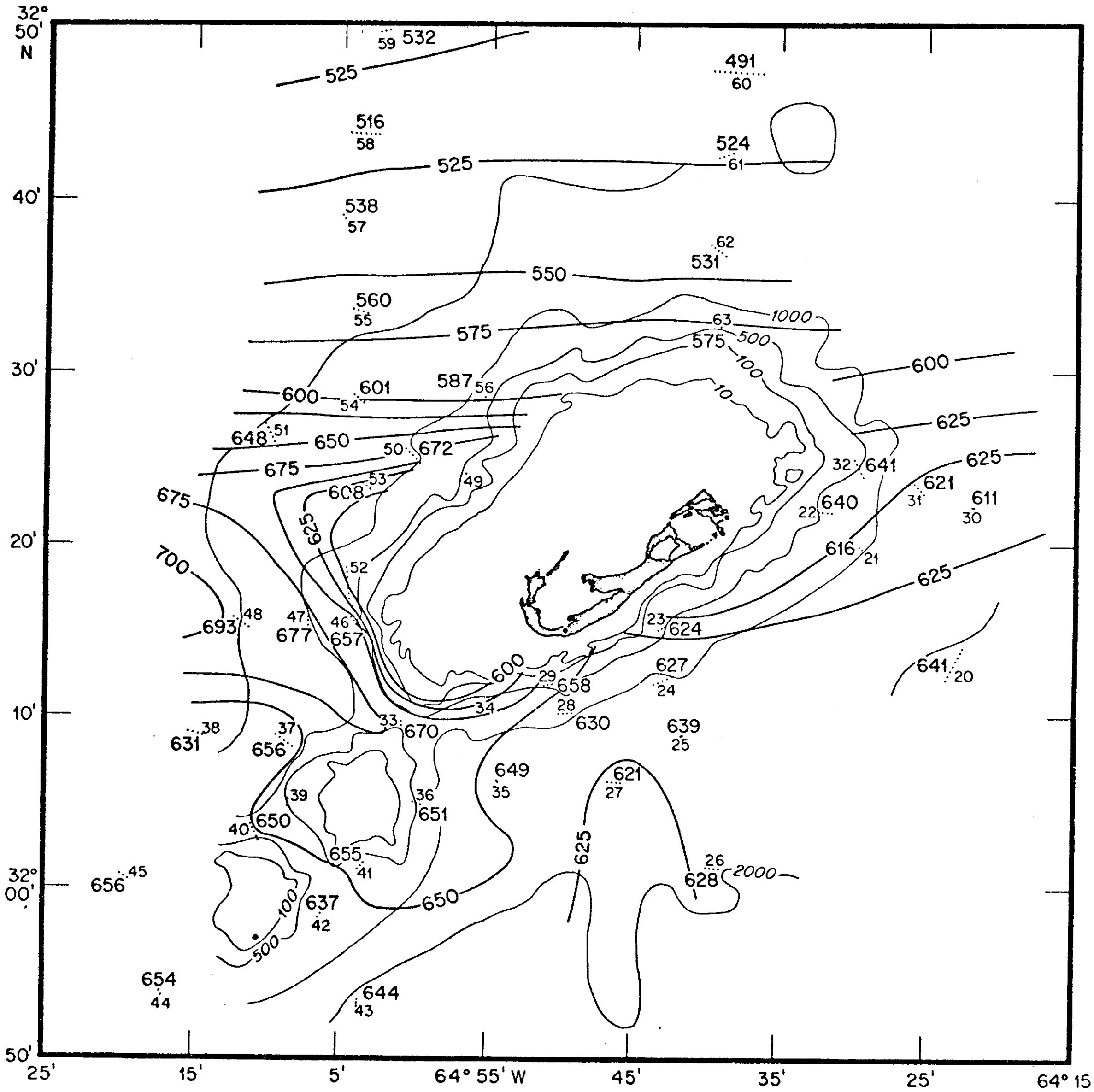


Figure 10. Depth to the 14°C isotherm in meters for GOSNOLD-144. Interpretation B.

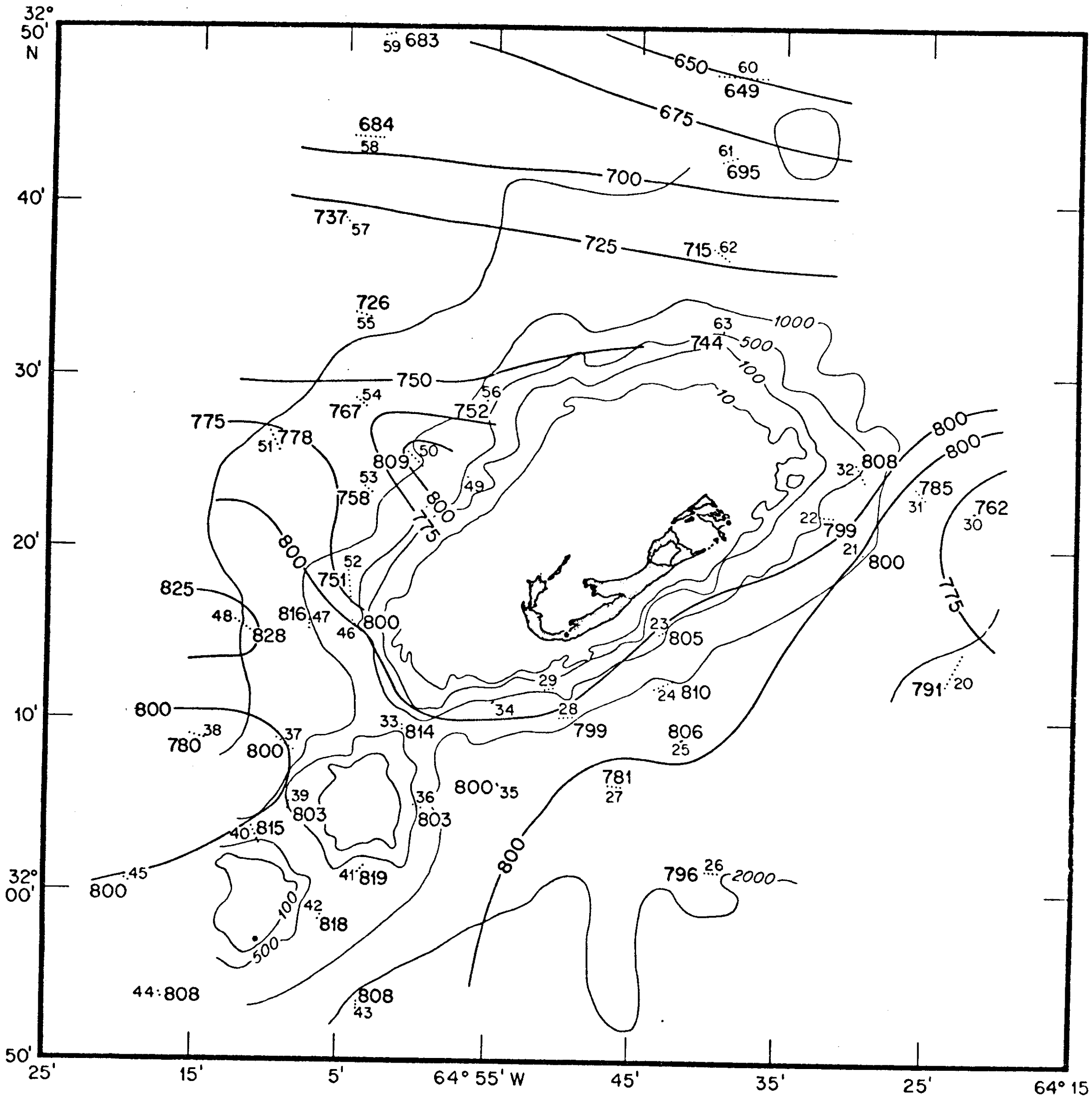



Figure 11. Depth to the 10°C isotherm in meters for GOSNOLD-144. Interpretation A.

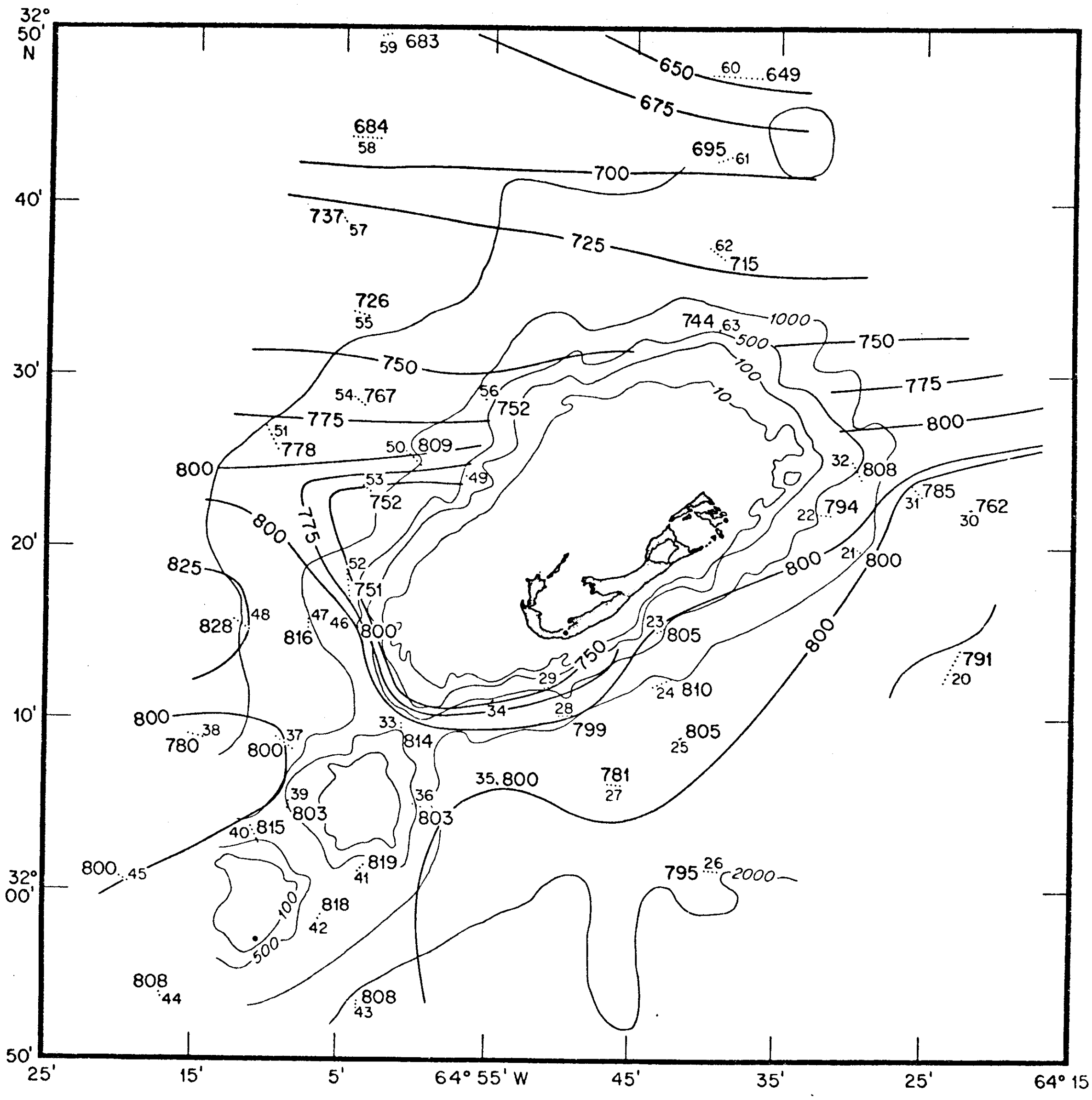



Figure 12. Depth to the 10°C isotherm in meters for GOSNOLD-144.
Interpretation B.

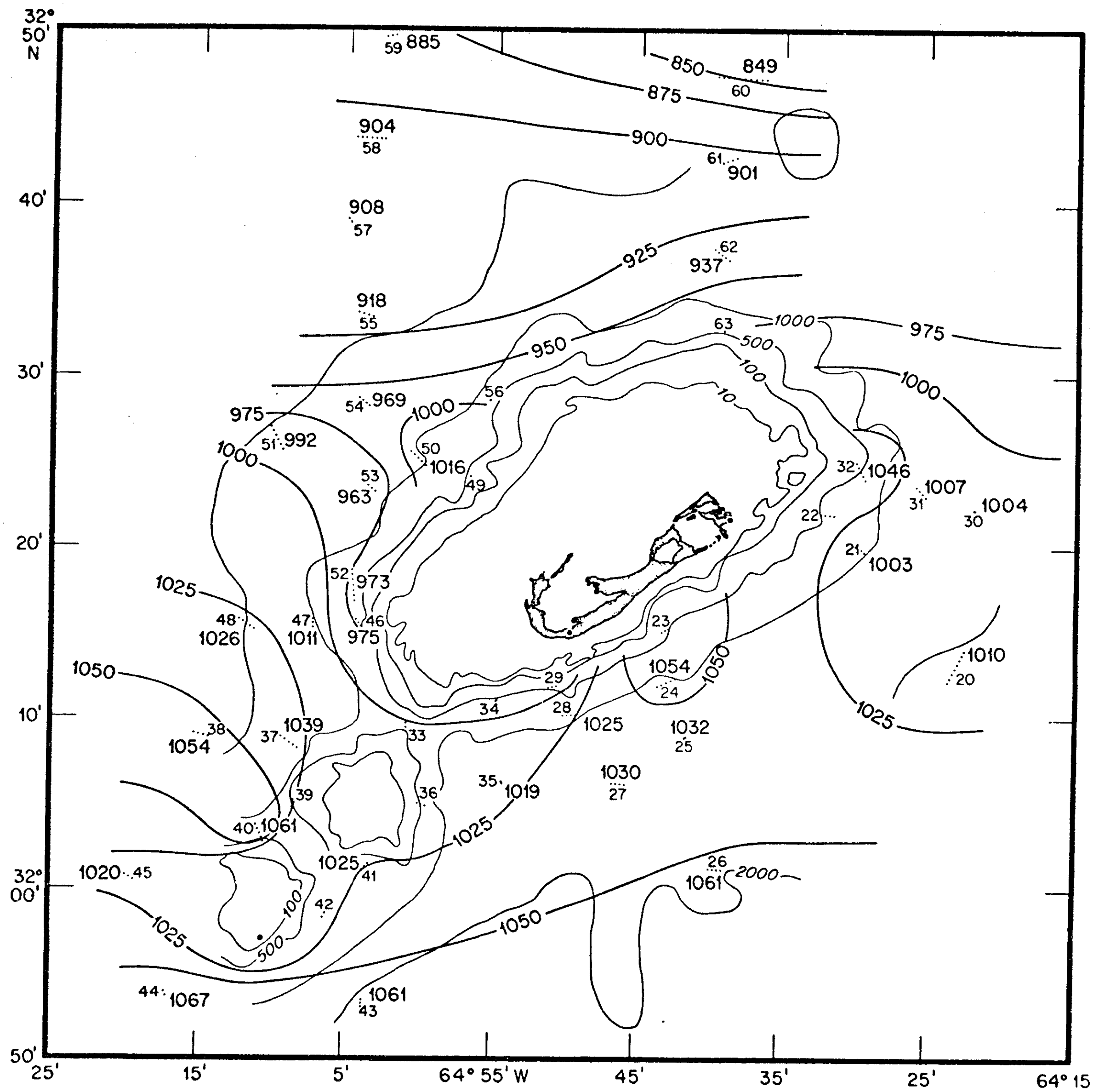



Figure 13. Depth to the 6°C isotherm in meters for GOSNOLD-144. Interpretation A.

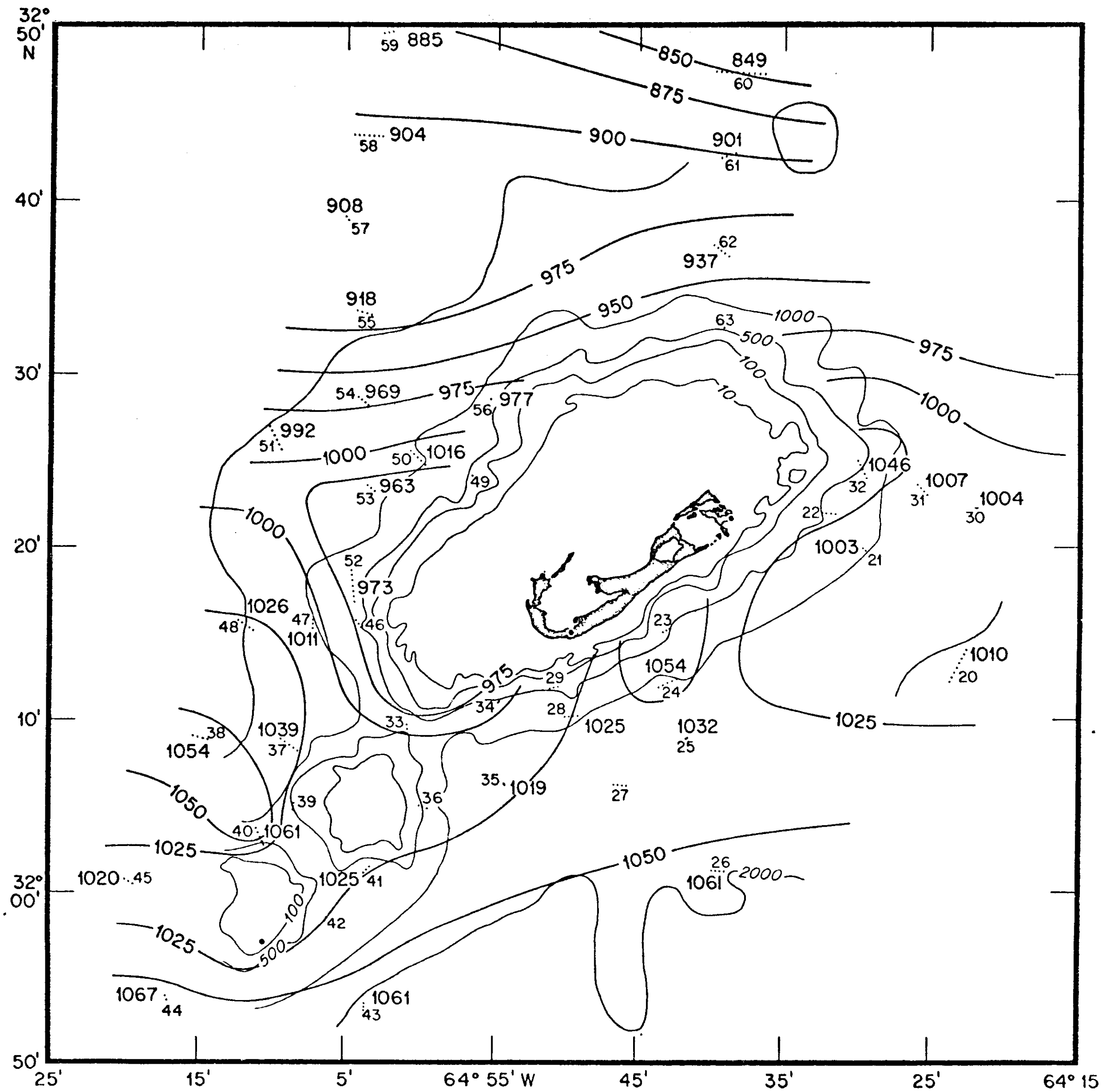



Figure 14. Depth to the 6°C isotherm in meters for GOSNOLD -144.
Interpretation B.

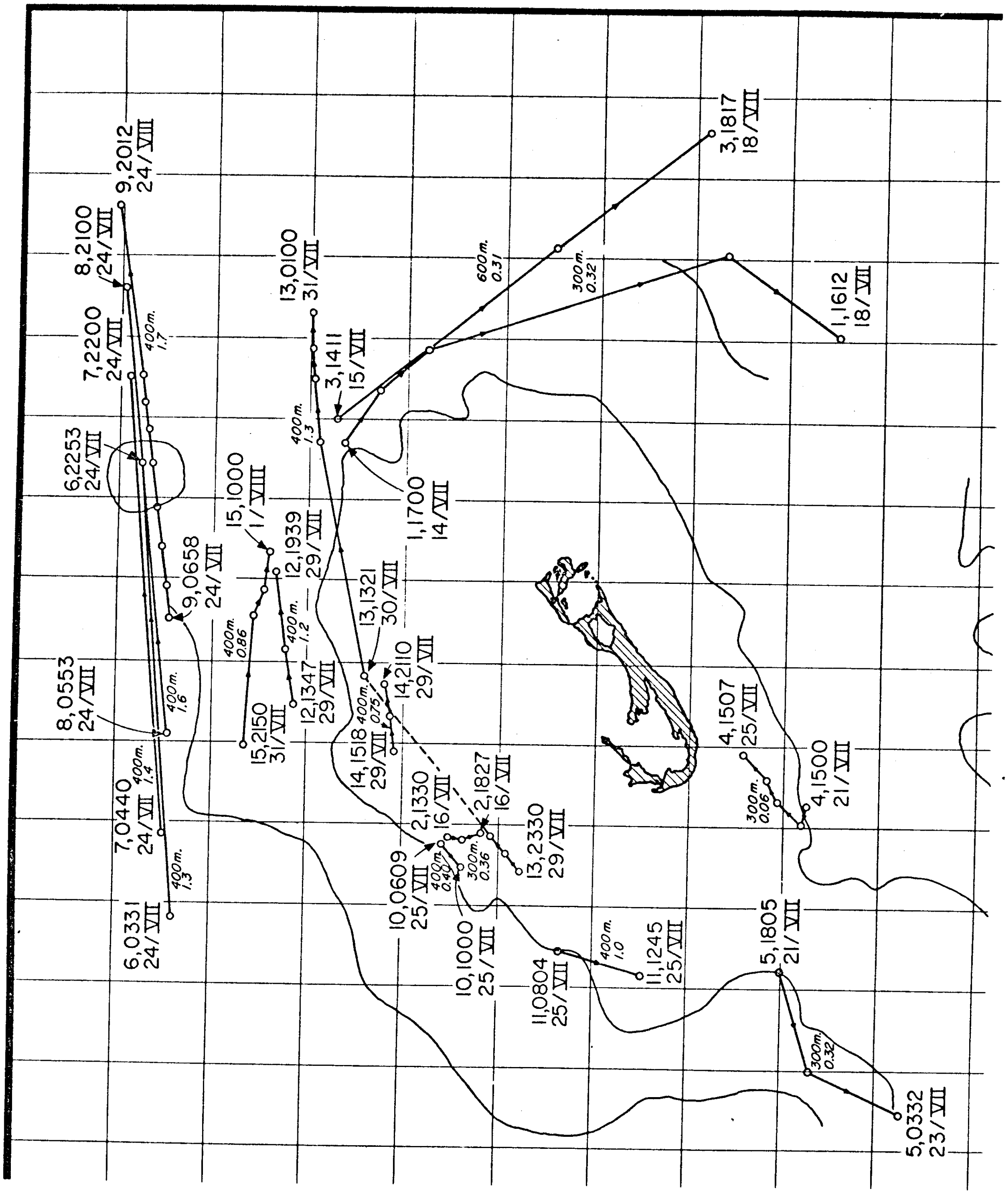



Figure 15. Drogue trajectories from GOSNOLD-144. The numbers at each location indicate drogue numbers, time of day, day and month respectively.

less to the east.

The thesis of this report is that these phenomena are manifestations of the adjustment that the ocean must make to the presence of the island. Two separate mechanisms are investigated and each is found to produce some agreement with observations made during different flow regimes. In chapter (2) the steady flow of an inviscid, stratified, rotating fluid around a circular island with steep sides is studied. A potential vorticity equation is found and analysed to first order in a small Rossby number and a small parameter measuring the angle that the island slope makes with the vertical. Although it is necessary to neglect circulation and simplify the problem arbitrarily, the calculated shape of the isotherms agrees well with one of the interpretations described above. The Gosnold data seems to be influenced by both slope and Rossby number effects while currents were smaller for the Atlantis 11 cruise and the island slope alone is most able to produce the observed distortion. Some speculations are made on the cause of the peculiar mixing areas.

Chapter (3) looks into the effects of nonlinearities in shoaling internal waves. The hypothesis is that a steady train of waves is incident on the northern coast and, while progressing up the beach, rectification of the wave motion occurs through the nonlinear advective terms and a tilt of the isotherms is forced in a manner analogous to the surface set-down forced by shoaling surface waves. In addition, a mean Eulerian current is produced that balances the Stokes drift inherent in the wave motion in such a way that the mean Lagrangian velocity vanishes and there is no net motion of particles into the beach.

Although the data is insufficient for an accurate computation of mean isotherm positions, a shift of the desired form is found in the earlier stations of Gosnold 144. The later stations do not conform to this pattern but are best explained by the steady flow hypothesis. Within the scope of the wave problem the mixing region can be produced by the breaking of the waves at some point on the beach.

# 2. STEADY FLOW PAST AN ISLAND

## 2.1. Introduction

The steady flow af an inviscid fluid past an obstacle is a classical problem in fluid dynamics. In addition to the traditional potential flow studies there is an increasing interest in the more geophysically relevant situations which include buoyancy and Coriolis forces. For the most part there have been two main lines of inquiry, the attack employed depending on the strength of the nonlinear advective terms in the equations of motion. If these nonlinearities are of the same order as the other terms in the inviscid equations then a system of stationary lee waves is found downstream from the obstacle whose wavelength and amplitude depend on the magnitude of the nonlinearities and the size of the obstacle. When advection can be considered small or large a steady flow pattern which resembles potential flow at the lowest order is established near the obstacle.

Lyra's (1946) approach to the O(1) regime for the nonlinear terms was to treat the obstacle as being a small deformation of the otherwise flat bottom and its perturbing effect on a uniform stream to be small. In this way he was able to linearize the problem in an Oseen manner and find a wave equation whose solutions were lee waves. His problem was two-dimensional but Scorer and Wilkinson (1956), using individual solutions to ridges inclined at angles to the stream, integrated over them to find the lee waves produced by an isolated hill. These waves were contained within a wedge behind the obstacle in a pattern resembling Kelvin ship waves forced by a body moving on a fluid interface. Crapper (1959,1962) has analysed the fully three-dimensional problem using the infinitesmal obstacle scheme to develop a linear problem. Using Fourier transform methods to solve the resulting wave equation, he is able to evaluate the inversion only asymptotically far from the obstacle. These approaches have all attempted to model the stratified (but not rotating) flow of air over mountains where a phenonemon resembling lee waves is observed.

The relaxation of the constraint on the size of the obstacle has proven difficult. Long (1953) derived a linear equation from the fully nonlinear equations for two-dimensional stratified (or rotating) flow past an object. Special assumptions had to be made about the upstream velocity and stratification but from this equation he was able to find finite amplitude lee waves behind finite obstacles which agreed well with experimental observations. This equation has stimulated a great deal of work in two-dimensional flows but no extensions to flows past three-dimensional, large obstacles have been discovered.

Lighthill (1960,1965,1967) in a series of papers has developed a method for obtaining information about the far field effects of steady currents around large disturbances in dispersive media, a technique which can be applied to rotating and stratified fluids. His approach is, once again, to linearize the equations with the Oseen approximation and then, after deriving the wave equation for the medium, to model the disturbance to the flow as an unknown forcing term. The equation can then be Fourier transformed and the inversion evaluated at asymptotically large distances from the obstacle using stationary phase methods in three-dimensions. The result is that only those waves are found at a given point which have a group velocity in that direction from the obstacle. In this way a wave "wake" is generated. Crapper's analysis is essentially of this nature but he is able to evaluate the forcing function by means of his small hill assumption. Keller and Munk (1970), using similar ideas, have calculated the phase line pattern in the wake behind an obstacle moving in a thermocline of finite thickness. Here a number of modes are generated, each with its own "ship wave" pattern for lines of constant phase. The unfortunate weakness of this method is that no information about the amplitude of the generated waves is determined when the disturbance such as an island has finite dimensions.

The situation is not completely hopeless. A disturbance in a steady stream will be most efficient at producing waves of its own scale or larger and inefficient at generating smaller scale motion. In the case of an island in an ocean of finite depth, low mode waves with horizontal wavelengths of the order of an island diameter are the most likely. At the same time the stationary waves have a Doppler shifted frequency of Uk where U is the current speed and k the wavenumber in the direction of U. For the waves to be in the inertial-gravity range the frequency must lie between the Coriolis frequency f and the Brunt-Vaisala frequency N or:

$$f \lesssim Uk \lesssim N.$$

In the Bermuda area $N \approx 2 \times 10^{-3}\ sec^{-1}$, $f = 7.7 \times 10^{-5} sec^{-1}$ and a typical current strength is 20 cm/sec. In terms of wavelength, $\lambda$, the above inequality becomes:

$$16km \gtrsim \lambda \gtrsim .3km$$

and it seems, therefore, that only small scale waves are compatible with the generating current and these waves are incompatible with the larger lengths associated with the island ($\sim$40km in diameter). Such waves could be forced by nonlinear processes near the island, which could have smaller scale structure, or by such topographic features as the seamounts found nearby. However, the conclusion is that the island is an inefficient source of inertial gravity waves of a lee wave nature.

These lee wave treatments, in the case of an island, are also unsatisfying because they yield solutions which are only valid asymptotically far from the disturbance. In order to deal with effects in the near field it is not possible to use an Oseen linearization of the equations as velocity perturbations are not small. In this regime another approach has been considered. Hawthorne and Martin (1955) considered a stream flowing past a hemisphere in which the vertical density variations were so small that the flow could be considered a small deviation from potential flow. The results of this small Richardson number flow were slightly extended by Drazin (1961) who also studied the high Richardson number case. Here buoyancy forces are strong enough to inhibit vertical motion and the lowest order flow is two-dimensional potential flow in each horizontal plane about the three-dimensional obstacle. Drazin was able to derive the secondary effects for both limits in the Richardson number.

It is noteworthy that these near field techniques exclude the lee wave phenomena. Both Rossby and Richardson numbers appear in the wavenumber expressions for the associated lee waves (as found by Lighthill's method) and this sinusoidal dependence is lost when limits of these parameters are taken in a perturbation scheme.

Neither the results of Hawthorne and Martin nor Drazin's are applicable to the island as Coriolis forces were neglected. Although the flow can be considered to be of high Richardson number the parameter relating buoyancy and Coriolis forces will be shown to be O(1). The analysis of this chapter attempts to account for this additional influence. A perturbation sequence in a small Rossby number is found to be appropriate. The lowest order is geostrophic and similar to three-dimensional potential flow in the scaled geometry even though the Richardson number is large. The Coriolis force, acting at right angles to the buoyancy force tends to return the fundamental flow to an isotropic force balance. Corrections at the next order to this flow are calculated with particular attention being paid to the effect on the isopycnals so that a comparison can be made with the data presented in chapter (1).

Inviscid flow problems, such as this is, have solutions which are charactaristically nonunique. An unknown function, which can be identified with circulation about the obstacle, enters the complete solution. Neither of the previously mentioned "near field" problems include this feature: both authors assume their flow to be symmetric about the axis of the steady upstream flow. In nonrotating situations with symmetric obstacles this assumption might be valid but, with a Coriolis force included, circulation cannot be dismissed in an obvious manner and its effects encumber the following analysis. In the O(1) flow the influence of this arbitrariness is accounted for but in the higher order problem it must be neglected.

The data is not sufficient to establish the validity of the zero circulation assumption, but the drogue tracks of figure (15) do not have a tendency to circle the island. Circulation has been observed around other islands such as Taiwan and Iceland where it has been ascribed by Stockmann (1966) to a combination of the wind curl and the relationship of the island to exterior boundaries. Wyrtki et al (1967,1970) have also observed circulation in the Hawaiian archipelago and Patzert (1969) has shown that it is driven in the inter-island channels by winds. These causes of circulation should be absent from an isolated island such as Bermuda. Longuet-Higgins (1967, 1969) and Rhines (1969) have shown that it is possible to trap low frequency oscillations which progress around islands. These effects have a zero mean (i.e. no net circulation) but Longuet-Higgins (1970) has also found both experimentally and theoretically that large scale inertial oscillations interacting with and island can produce a mean streaming around the island. He uses these ideas to explain some buoy tracks in the Bermuda area presented by Stommel (1954) which have indications of this tendency. The following analysis assumes that any time dependent processes are vanishingly small compared to inertial forces and therefore the features produced by them can be ignored.

Finally there is the question of whether or not streamlines separate from the viscous boundary layer on the obstacle to form a wake downstream. Section (2.5) investigates the effects of rotation on this phenomenon and the conclusion is reached that separation might not occur in the high Reynolds number flow provided that the Rossby number is small enough. Recent experimental work by Boyer (1970) supports this conclusion.

## 2.2 Formulation

Consider the steady flow of a stratified, incompressible fluid in a rectangular Cartesian coordinate system which rotates about the z-axis in such a way that the Coriolis parameter $f=f_o+\beta y$. The velocity $\vec{u}(x,y,z)$ has components $(u,v,w)$ depending on coordinates $(x,y,z)$ where the x-axis is to the east, the y-axis to the north, and the z-axis vertical, parallel to the rotation vector and antiparallel to the force of gravity $-g\hat{k}$. The fluid pressure is $p(x,y,z)$ and the density $\rho(T)$ where T is a state variable chosen to be temperature for convenience. The dependence of $\rho(T)$ on T is approximated by a linear relation with the proportionality constant $\alpha$ being the coefficient of thermal expansion for the fluid. $\nu$ and $K$ are the coefficients of kinematic viscosity and heat conductivity respectively. Under these restrictions and definitions the equations of motion are:

$$\vec{u}\cdot\nabla\vec{u} - f\hat{k}\times\vec{u} = -\frac{\nabla p}{\rho} - g\hat{k} + \nu\nabla^2\vec{u} \tag{2.1}$$

$$\nabla\cdot\vec{u} = 0 \tag{2.2}$$

$$\vec{u}\cdot\nabla T = K\nabla^2 T \tag{2.3}$$

$$\rho = \rho_o\left(1-\alpha\left(T-T_o\right)\right) \tag{2.4}$$

Rigid horizontal planes at $z=0$ and $z=H$ and a rigid obstacle, typified by a horizontal scale L and centered at the origin bound the fluid. The steady motion, far from the obstacle, is assumed to be horizontally uniform with velocity components $(u_o(z),v_o(z),0)$ having a typical scale U. The basic stratification - that temperature variation in the fluid that would exist in the absence of motion - is taken to be a linear function of z with a bottom temperature $T_o$ and a temperature difference between top and bottom of $\Delta T$. After employing the Boussinesq approximation to neglect all stratification effects except those coupled directly with gravity, the variables are scaled with the intention of balancing the linear terms in (2.1) to (2.4) and having the nonlinear ones multiplied by nondimensional parameters.

From these considerations emerge a number of parameters. The aspect ratio $\delta = H/L$ relates the vertical and horizontal scales; the Rossby number $\epsilon = U/fL$ determines the relative importance of inertial and Coriolis forces; the Ekman number $E = \nu/fH^2$ is a measure of the strength of viscous against Coriolis forces; a stratification parameter $S = Hg\alpha\Delta T/L^2f^2$ indicates the relative importance of buoyancy and Coriolis forces in the scaled geometry; the Prandtl number $\sigma = \nu/\kappa$ is the ratio of the coefficients of viscosity and thermal diffusion; and finally a parameter $\beta' = \beta L/f_o$ measures the effect of changes in the Coriolis parameter over the horizontal scale of motion. Letting primes indicate scaled, nondimensional quantities, the following procedure was used:

$$\begin{aligned}
(u,v,w) &= U(u',v',\delta w') \\
(x,y,z) &= L(x',y',\delta z') \qquad\qquad (2.5) \\
p &= p_o + p_s(z') + \epsilon\rho_o f^2L^2p'(x',y',z') \\
T &= T_o + \Delta T z' + \Delta T(\epsilon/S)T'(x',y',z').
\end{aligned}$$

$p_o$ is chosen to be the pressure at $z'=0$ produced by the basic stratification while $p_s(z')$ is the hydrostatic pressure resulting from the balance:

$$\frac{\partial}{\partial z'} p_s(z') = -g\rho_o H + g\alpha\rho_o H \Delta T z'$$

Ocean temperature structure is, of course, not a linear function of z and a more accurate model should include a normalized function of z in the temperature perturbation scheme (i.e. $T=T_o + \Delta T T_s(z') + ---$). If this function varies slowly enough then the results of this analysis can be regarded as the lowest order in a WKB perturbation. To obtain more accurate quantitative results S can be replaced by $ST_s(z')/z'$ in the final expressions.

The above scaling procedure , substituted into equations (2.1) to (2.4), yields the following set after the Boussinesq approximation has been made:

$$\epsilon\, \vec{u}'\cdot\nabla' u' - (1+\beta' y') v' = -p'_{x'} + E \frac{\partial^2 u'}{\partial z'^2} + \delta^2 E \nabla_h'^2 u' \quad (2.6)$$

$$\epsilon\, \vec{u}'\cdot\nabla' v' + (1+\beta' y') u' = -p'_{y'} + E \frac{\partial^2 v'}{\partial z'^2} + \delta^2 E \nabla_h'^2 v' \quad (2.7)$$

$$\delta^2\epsilon\, \vec{u}'\cdot\nabla' w' = -p'_{z'} + T' + \delta^2 E \frac{\partial^2 w'}{\partial^2 z'^2} + \delta^4 E \nabla_h'^2 w' \quad (2.8)$$

$$\epsilon\, \vec{u}'\cdot\nabla' T' + S\, w' = \frac{E}{\sigma} \frac{\partial^2 T'}{\partial z'^2} + \delta^2 \frac{E}{\sigma} \nabla_h'^2 T' \quad (2.9)$$

$$\nabla'\cdot\vec{u}' = 0 \quad (2.10)$$

In order to proceed further in the analysis, further approximations must be made. Taking the flat bottom to be at H=1.5km and using the island diameter at this depth to be the horizontal length scale L=45 km, then the aspect ratio $\delta$ =.033. Note that the aspect ratio occurs in the equations as the square and

$\delta^2=1.0\times10^{-3}$. Avelocity scale for the Gosnold 144 cruise is of the order U=40 cm/sec and with $f=7.7\times10^{-5}$ the Rossby number $\epsilon=.12$ which is considerably larger than $\delta^2$. The stratification parameter S can be rewritten in terms of the Brunt-Vaisala frequency $N=(g\alpha\Delta T/H)^{1/2}$ as:

$$S = \left(\frac{\delta N}{f}\right)^2. \tag{2.11}$$

A good value for N is $N=2.5\times10^{-3}$ $\sec^{-1}$giving a value of S=1.1 which is to be considered O(1) compared to the other parameters. A value for the viscosity to be used in computing an Ekman number can only be guessed at but using the other scales $E=\nu/2.1\times10^9$ and for most reasonable guesses $\epsilon > E$. Indeed the Ekman number is probably sufficiently small that $\epsilon > E^{1/2}$ meaning that typical vertical velocities produced by the advection of the temperature field according to equation (2.9) are much greater than those produced by Ekman layer suction and diffusive effects are negligible outside of thin boundary layers. $\beta'=4.4\times10^{-3}$ indicating that very small changes in the Coriolis parameter occur over scales of length L. The conclusion is that $\beta'$, $\delta^2$, and E are all much smaller than $\epsilon$ and in order to isolate the more important effects caused by nonlinear advection these parameters can be set equal to zero to give the following set of equations: (dropping the primes)

$$\epsilon\vec{u}\cdot\nabla u - v = -p_x \tag{2.12}$$

$$\epsilon\vec{u}\cdot\nabla v + u = -p_y \tag{2.13}$$

$$0 = -p_z + T \tag{2.14}$$

$$\epsilon \vec{u}\cdot\nabla T + S w = 0 \qquad (2.15)$$

$$\nabla\cdot\vec{u} = 0 \qquad (2.16)$$

With the removal of terms depending on the variation in the Coriolis parameter the coordinate axis can be rotated about the z-axis without changing the above set in such a way that the upstream flow is directed along the positive x-axis with speed $u_o(z)$. If $\beta$ effects were included such a uniform flow could not exist as it does not conserve the full potential vorticity which includes a planetary vorticity contribution. Within this reasoning the uniform conditions used must be assumed to be a local approximation to some larger scale motion, such as baroclinic Rossby waves, which include planetary vorticity effects.

If pressure is eliminated from (2.12) and (2.13) one finds that:

$$\epsilon \vec{u}\cdot\nabla\zeta = \epsilon\zeta w_z + w_z \qquad (2.17)$$

where $\zeta = v_x - u_y$ is the vertical component of vorticity. Substitution from (2.15) for $w_z$ gives:

$$\vec{u}\cdot\nabla\left(\zeta + \frac{T_z}{S}\right) = \zeta w_z - \vec{u}_z\cdot\nabla\left(\frac{T}{S}\right) \qquad (2.18)$$

The quantity $\zeta + T_z/S$ is a potential vorticity (without the planetary vorticity contribution) and equation (2.18) describes the changes in this vorticity along streamlines brought about by vortex line stretching through variations in the vertical velocity ( $\zeta w_z$ ) and separation of the isotherms ($-\vec{u}_z.\nabla T/S$).

On the top and bottom boundaries the inviscid constraint is that the vertical velocity vanish. This implies that:

$$w = \vec{u}\cdot\nabla T = 0 \quad \text{on} \quad z = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.19}$$

from equation (2.15). The condition at the island is that there be no normal velocity component. If the island is described by the relation $z=f(x,y)$ and the normal $\vec{n}=\hat{k}-\nabla_h.f$ then:

$$\vec{u}\cdot\vec{n} = w - \vec{u}_h\cdot\nabla_h f = 0 \quad \text{on} \quad z = f(x,y). \tag{2.20}$$

To solve the set (2.12) to (2.16) using the vorticity equation (2.18) and the boundary conditions (2.19) and (2.20) the mathematical procedure will be to expand all variables in a Rossby number power series of the form:

$$\vec{u}(x,y,z;\epsilon) = \sum_{n=0}^{\infty} \epsilon^n \vec{u}^{(n)}(x,y,z) \tag{2.21}$$

Substitution of these series into the equations gives rise to a series of problems, one at each order in $\epsilon$ . As in homogeneous, nonrotating flow problems where viscosity is neglected the solutions will exhibit a certain arbitrariness that can be attributed to an unknown circulation around the island. Even though this is a high Reynolds number flow it is assumed that another phenonemon of the nonrotating problem, namely detachment of the boundary layer streamlines in the rear of the obstacle to form a wake, does not occur. If this were to happen, the solutions making use of inviscid boundary conditions applied at the solid surfaces would be invalid. Further comments on this problem will be made in section (2.6).

## 2.3. The O(1) Problem - Slope Effects

Using a perturbation expansion for each variable of the form of equation (2.21) and keeping only the $O(\epsilon^0)$ terms, equations (2.12) to (2.16) become:

$$-v^{(0)} = -p_x^{(0)} \tag{2.22}$$

$$u^{(0)} = -p_y^{(0)} \tag{2.23}$$

$$0 = -p_z^{(0)} + T^{(0)} \tag{2.24}$$

$$w^{(0)} = 0 \tag{2.25}$$

$$u_x^{(0)} + v_y^{(0)} + w_z^{(0)} = 0 \; . \tag{2.26}$$

To this order, the flow around the obstacle is horizontal and geostrophic. Because $w^{(0)}=0$ continuity allows the definition of a streamfunction $\psi(x,y,z)$ according to:

$$u^{(0)} = -\psi_y$$
$$v^{(0)} = \psi_x \; . \tag{2.27}$$

From the horizontal momentum equations (2.22) and (2.23), the streamfunction can be made identical with the pressure $p^{(0)}$ by defining the arbitrary function of z in the integration to be zero. Therefore:

$$\psi \equiv p^{(0)}, \tag{2.28}$$

and

$$T^{(0)} = \psi_z \; . \tag{2.29}$$

Equation (2.18), in the streamfunction variable, reduces to:

$$-\psi_y \frac{\partial}{\partial x}\left(\nabla_h^2\psi + \frac{\psi_{zz}}{S}\right) + \psi_x \frac{\partial}{\partial y}\left(\nabla_h^2\psi + \frac{\psi_{zz}}{S}\right) = 0$$

If the Jacobian determinant of two functions f(x,y,z) and g(x,y,z) with respect to x and y is defined by:

$$J(f,g) = f_x g_y - f_y g_x , \tag{2.30}$$

this vorticity relation can be put in the form:

$$J\left(\psi, \nabla_h^2\psi + \frac{\psi_{zz}}{S}\right) = 0$$

and then integrated to give:

$$\nabla^2\psi + \frac{\psi_{zz}}{S} = H_1(\psi) . \tag{2.31}$$

$H_1(\psi)$ is an as yet undetermined function which will be evaluated from the uniform flow conditions found upstream from the obstacle. As the x-axis is aligned with this flow, equations (2.22) and (2.23) asymptotically become:

$$0 = -p_x^{(0)} = -\psi_x \tag{2.32}$$

$$u_0(z) = -p_y^{(0)} = -\psi_y .$$

Integration yields :

$$\psi \approx -u_0(z)\, y \quad \text{as} \quad x \to \infty , \tag{2.33}$$

where the arbitrary function of z in the integration has been chosen to vanish so that $\psi=0$ on y=0 far from the obstacle. This also establishes the zero of the pressure perturbation to be at y=0 upstream. The unknown function $H_1(\psi)$ in (2.31) can now be found by substituting (2.33) into (2.31) to get:

$$H_1(\psi) = \nabla_h^2\psi + \frac{\psi_{zz}}{S} = -\frac{u_o''(z)}{S}y$$
$$= \frac{u_o''(z)}{u_o(z)} \cdot \frac{\psi}{S} .$$

The streamfunction equation (2.31) becomes:

$$\nabla^2\psi + \frac{\psi_{zz}}{S} = \frac{u_o''(z)}{S\,u_o(z)}\,\psi . \qquad (2.34)$$

The term on the right hand side of this equation has a coefficient which is a function of z and will cause problems in an analytic solution unless the upstream flow function $u_o(z)$ is given an appropriate form. If $u_o(z)$ varies trigonometrically or hyperbolically then $u''(z)/u\ (z)$ is a constant but the simplest choice is the linear dependence:

$$u_o(z) = a + b.z, (a,b \text{ constants}) \qquad (2.35)$$

in harmony with the simple form assumed for the stratification. Equation (2.34), except for a scale factor, reduces to Laplace's equation:

$$\nabla_h^2\psi + \frac{\psi_{zz}}{S} = 0 . \qquad (2.36)$$

This statement is an expression of the fact that potential vorticity is conserved in the $O(\epsilon^0)$ motion.

The inviscid boundary conditions on the rigid walls are that the normal velocities vanish. On top and bottom this implies that the vertical velocity vanishes. To $O(\epsilon^0)$ $w^{(0)}=0$ and the condition is automatically satisfied but the condition on the streamfunction comes from the $O(\epsilon)$ heat equation which becomes (2.19) when evaluated at the boundary and gives:

$$u^{(0)}\cdot\nabla T^{(0)} = 0 \quad \text{on} \quad z = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.37}$$

In terms of the streamfunction one finds that:

$$J(\psi, \psi_z) = 0$$

or

$$\psi_z = H_2(\psi) \quad \text{on} \quad z = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Once again the integration function $H_2(\psi)$ is found from application of condition (2.23) away from the obstacle. This gives:

$$\psi_z = \frac{u_0'(1)}{u_0(1)}\psi = \frac{b}{a+b}\psi \quad \text{on } z = 1 \tag{2.38}$$

$$= \frac{u_0'(0)}{u_0(0)}\psi = \frac{b}{a}\psi \quad \text{on } z = 0. \tag{2.39}$$

At the island boundary condition (2.20) yields:

$$u^{(0)}f_x + v^{(0)}f_y = 0$$

or

$$J(\psi, f) = 0 \qquad \text{on} \quad z = f(x,y).$$

to $O(\epsilon^0)$ provided that $|\nabla_h f| > \epsilon$. Therefore:

$$\psi = G(f) = \bar{G}(z) \qquad \text{on} \quad z = f(x,y) \tag{2.40}$$

where G(z) is an undetermined function.

G(z) is, in fact, indeterminant in this inviscid theory and is directly related to the arbitrary circulation effect found in inviscid, homogeneous, nonrotating flow problems. If the flow is to be symmetrical in the y direction (cross stream symmetry) then the obstacle must also possess this symmetry and the x-axis is the streamline $\psi = 0$ which intersects the obstacle and forces G(z) to vanish. For $G(z) \neq 0$ the flow will not have this symmetry.

The vorticity equation (2.36) with conditions (2.33), (2.38), (2.39) and (2.40) defines the problem. In order to employ a separation of variables technique the geometry of the island must be simplified somewhat. A model which suits Bermuda well is expressed by the dimensional equation:

$$z = \gamma \left( \frac{L}{2} - r \right)$$

where $\gamma$ is the average island slope (i.e. the average value of $\nabla_h \cdot f(x,y)$) and radial coordinates are defined by: (see fig.16)

$$\begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta . \end{aligned} \tag{2.41}$$

In scaled form this geometry becomes:

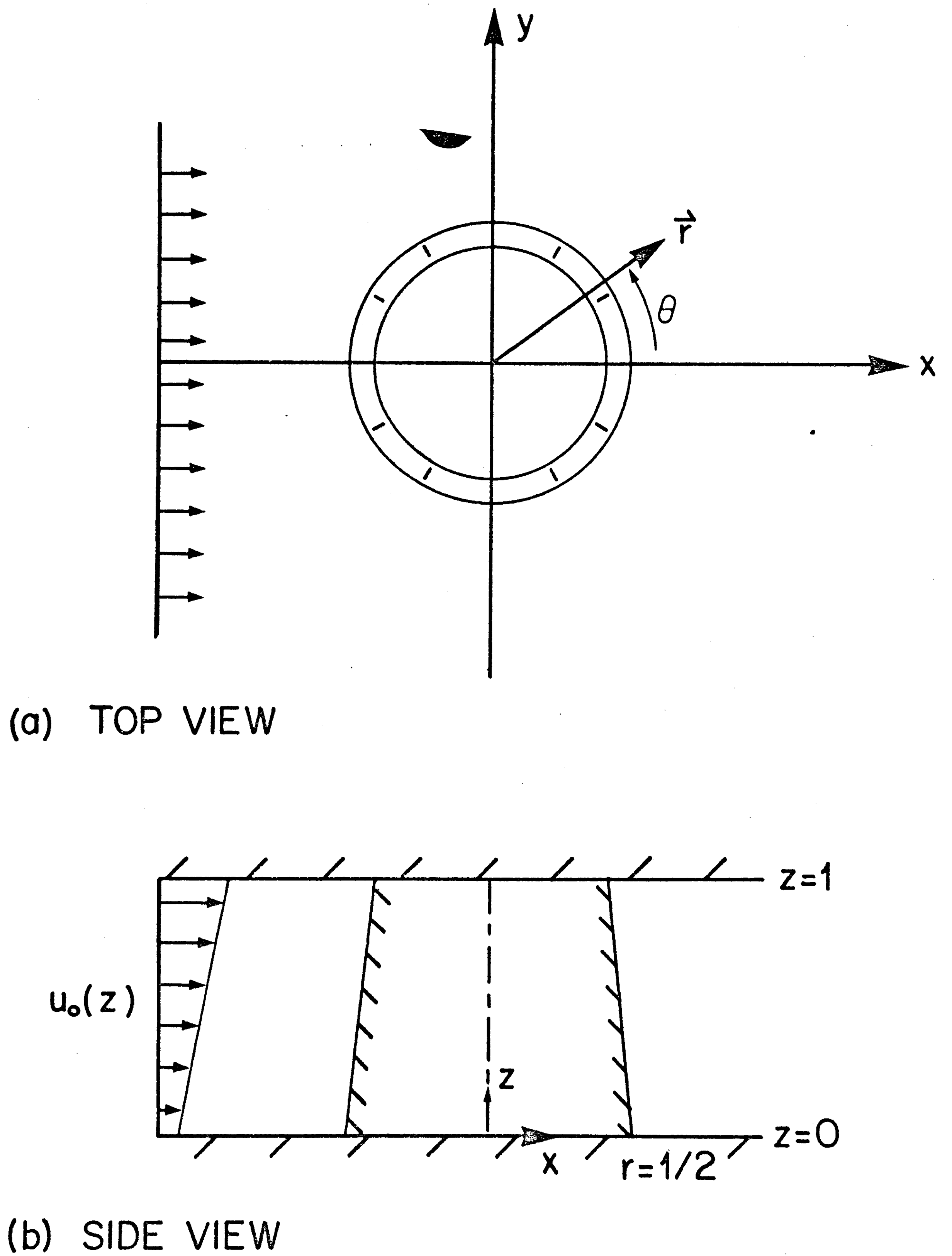



Figure 16. Flow - obstacle configuration.

$$z' = \frac{\gamma}{\delta}\left(\frac{1}{2} - r'\right)$$

or

$$r' = \frac{1}{2} - \frac{\delta}{\gamma} z'.$$

The Bermuda slope is relatively steepaveraging $\gamma$ =.25 down to 2km depth. With the previously calculated value of the aspect ratio one finds that:

$$r = \frac{1}{2} - .13\, z. \tag{2.42}$$

Letting this scaled slope, which is in reality a "co-slope", be denoted by the parameter $\alpha$ =.13 the conclusion is that the island deviates very little ( at least in the vertical) from a right circular cylinder. Effects of the slope can be incorporated by a new perturbation series in $\alpha$ and a representation of the boundary conditions at the island by equivalent ones at $r=\frac{1}{2}$ through a Taylor series expansion. That is:

$$G(z) = \psi(r=\tfrac{1}{2}-\alpha z, z) = \psi(r=\tfrac{1}{2}, z) - \alpha z \left.\frac{\partial\psi}{\partial r}\right|_{r=\frac{1}{2}} + O(\alpha^2) \tag{2.43}$$

In polar coordinates the upstream condition (2.33) becomes:

$$\psi \rightsquigarrow -U_0(z)\, r \sin\theta \quad \text{as } r \rightsquigarrow \infty. \tag{2.44}$$

and this introduces the only explicit angular dependence in the problem, namely $\sin\theta$ . This dependence can be separated by the substitution:

$$\psi(r,\theta,z;\alpha) = F_0(r,z;\alpha) + F_1(r,z;\alpha)\sin\theta \tag{2.45}$$

which generates two problems:

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}F_m - \frac{m^2}{r}F_m + \frac{1}{S}\frac{\partial^2 F_m}{\partial z^2} = 0\,, \quad (m=0,1) \tag{2.46}$$

with

$$\left.\begin{aligned} F_0 &\rightsquigarrow 0 \\ F_1 &\rightsquigarrow -u_0(z)\,r \end{aligned}\right\} \text{as } r \rightsquigarrow \infty\,, \tag{2.47}$$

$$\begin{aligned} \frac{\partial F_m}{\partial z} &= \frac{u_0'(1)}{u_0(1)}F_m \quad \text{on } z=1\,, \\ &= \frac{u_0'(0)}{u_0(0)}F_m \quad \text{on } z=0\,, \end{aligned} \tag{2.48}$$

and

$$\begin{aligned} F_0 &= G(z) \\ F_1 &= 0 \end{aligned} \quad \text{on } r = \tfrac{1}{2} - \alpha z. \tag{2.49}$$

The arbitrariness defined as circulation is involved only in the $F_0(r,z;\alpha)$ function and, for $G(z)=0$, only the trivial solution $F_0 \equiv 0$ is allowed.

Expanding each variable in a power series in the small parameter $\alpha$ according to:

$$F_m(r,z;\alpha) = \sum_{n=0}^{\infty} \alpha^n F_m^{(n)}(r,z) \qquad (m=0,1) \tag{2.50}$$

a new problem sequence is generated. As $\alpha$ does not appear explicitly in the differential equation (2.46) an identical equation will describe the flow at each order in $\alpha$. The same is true for the boundary conditions at z=0 and z=1. Using the Taylor series expansion in (2.43):

$$\left.\begin{aligned} F_0^{(0)}(\tfrac{1}{2},z) &= G(z) \\ F_1^{(0)}(\tfrac{1}{2},z) &= 0 \\ F_m^{(1)}(\tfrac{1}{2},z) &= z\left.\frac{\partial F_m^{(0)}}{\partial r}\right|_{r=\frac{1}{2}} \end{aligned}\right\} \text{at } r=\tfrac{1}{2}\,, \tag{2.51}$$

and
$$\left.\begin{array}{l} \dfrac{F_1^{(0)}}{r} \rightsquigarrow -u_0(z) \\ \dfrac{F_0^{(n)}}{r} \text{ and } \dfrac{F_0^{(1)}}{r} \rightsquigarrow 0 \end{array}\right\} \text{as } r \rightsquigarrow \infty . \; (n=0,1\cdots) \qquad (2.52)$$

The r and z dependences can now be separated from the problems posed above. An eigenvalue problem is found in the vertical and the r variation is contained in solutions to a modified Bessel equation. To $O(\alpha^0)$:

$$F_0^{(0)}(r,z) = A\,u_0(z) + C_0\,u_0(z)\ln r + \sum_{\ell=1}^{\infty} C_\ell\, u_\ell(z)\, K_0\!\left(\frac{k_\ell r}{\sqrt{S}}\right),$$
$$F_1^{(0)}(r,z) = -u_0(z)\left(r - \frac{1}{4r}\right), \qquad (2.53)$$

where
$$u_\ell(z) = b \sin k_\ell z + a k_\ell \cos k_\ell z \quad (\ell = 1,2,\cdots) \qquad (2.54)$$

is the vertical eigenfunction and $k_\ell$ is the $\ell^{th}$ largest root of the eigenvalue equation:

$$\tan k_\ell = \frac{b^2 k_\ell}{(a+b)a k_\ell^2 + b^2} . \qquad (2.55)$$

A can be taken to be zero because it has no effect on the velocities and represents a constant shift in the isotherms. The other constants are defined by:

$$C_0 = \frac{\int_0^1 G(z)\, u_0(z)\, dz}{\ln\frac{1}{2} \int_0^1 u_0^2(z)\, dz}, \qquad (2.56)$$

and
$$C_\ell = \frac{\int_0^1 G(z)\, u_\ell(z)\, dz}{K_0\!\left(\frac{k_\ell}{2\sqrt{S}}\right) \int_0^1 u_\ell^2(z)\, dz} . \qquad (2.57)$$

The streamfunction to $O(\alpha^0)$ has the form:

$$\psi(r,\theta,z) = C_o u_o(z) \ln r + \sum_{\ell=1}^{\infty} C_\ell u_\ell(z) K_o\left(\frac{k_\ell r}{\sqrt{S}}\right) - u_o(z)\left(r - \frac{1}{4r}\right)\sin\theta + O(\alpha) \tag{2.58}$$

Taking the radial derivative to find the azimuthal velocity, the circulation, C(r,z), can be found by integrating this quantity around a circle of radius r centered at the origin. In this way:

$$C(r,z) = \int_0^{2\pi} \psi_r \, r \, d\theta$$
$$= 2\pi\left\{C_o u_o(z) + \sum_{\ell=1}^{\infty} r C_\ell u_\ell(z) K_o'\left(\frac{k_\ell}{2\sqrt{S}}\right)\frac{k_\ell}{\sqrt{S}}\right\} \tag{2.59}$$

The circulation vanishes for all r and z only if $C_0$ and $C_1$ vanish which is true only if $G(z) \equiv 0$. Note that the circulation need not be constant in this stratified problem.

The $O(\alpha)$ boundary conditions on the island are found by substituting the $O(\alpha^o)$ solutions into (2.51) to get:

$$F_o^{(1)}\left(\tfrac{1}{2}, z\right) = z\left(2 C_o u_o(z) + \sum_{\ell=1}^{\infty} C_\ell u_\ell(z) \frac{k_\ell}{\sqrt{S}} K_o'\left(\frac{k_\ell}{2\sqrt{S}}\right)\right)$$
$$= Z_o(z), \tag{2.60}$$

and

$$F_1^{(1)}\left(\tfrac{1}{2}, z\right) = z\left[-2 u_o(z)\right]$$
$$= Z_1(z) \tag{2.61}$$

Again using separation of variables the solutions are found to be:

$$F_o^{(1)}(r,z) = C_o^{(1)} u_o(z) \ln r + \sum_{\ell=1}^{\infty} C_\ell^{(1)} u_\ell(z) K_o\left(\frac{k_\ell r}{\sqrt{S}}\right), \tag{2.62}$$

and

$$F_1^{(1)}(r,z) = \frac{B_o u_o(z)}{r} + \sum_{\ell=1}^{\infty} B_\ell u_\ell(z) K_1\left(\frac{k_\ell r}{\sqrt{S}}\right), \tag{2.63}$$

with

$$C_o^{(1)} = \frac{\int_0^1 Z_o(z) u_o(z)\, dz}{\ln\frac{1}{2} \int_0^1 u_o^2(z)\, dz}, \tag{2.64}$$

$$C_\ell^{(1)} = \frac{\int_0^1 Z_0(z)\, u_\ell(z)\, dz}{K_0\left(\frac{k_\ell}{2\sqrt{S}}\right) \int_0^1 u_\ell^2(z)\, dz}, \qquad (2.65)$$

$$B_0 = \frac{\int_0^1 Z_1(z)\, u_0(z)\, dz}{2\int_0^1 u_0^2(z)\, dz}, \qquad (2.66)$$

and

$$B_\ell = \frac{\int_0^1 Z_1(z)\, u_\ell(z)\, dz}{K_1\left(\frac{k_\ell}{2\sqrt{S}}\right) \int_0^1 u_\ell^2(z)\, dz}. \qquad (2.67)$$

What these solutions mean in terms of the perturbation temperature fieldcan be found from equation (2.24) and the knowledge that $\Psi = p^{(0)}$. Therefore:

$$T^{(0)} = C_0 u_0'(z) \ln r + \sum_{\ell=1}^{\infty} C_\ell u_\ell'(z) K_0\left(\frac{k_\ell r}{\sqrt{S}}\right) - u_0'(z)\left(r - \frac{1}{4r}\right)\sin\theta$$
$$+ \alpha\left\{ C_0^{(1)} u_0'(z) \ln r + \sum_{\ell=1}^{\infty} C_\ell^{(1)} u_\ell'(z) K_0\left(\frac{k_\ell r}{\sqrt{S}}\right) \right.$$
$$\left. + \left(\frac{B_0 u_0'(z)}{r} + \sum_{\ell=1}^{\infty} B_\ell u_\ell'(z) K_1\left(\frac{k_\ell r}{\sqrt{S}}\right)\right)\sin\theta \right\} + O(\alpha^2) \qquad (2.68)$$

With this result the shape of the isotherms can be derived from the scaling procedure in equation (2.5).

Although expression (2.68) is involved the physics is rather simple. The $O(\epsilon^0)$ flow is both hydrostatic and geostrophic and therefore satisfies the thermal wind balance. As the fluid is forced around the sides of the island the shear is altered and the temperature field adjusts. Consider the case having no circulation $(G(z)=0)$ so that:

$$\Psi = -u_0(z)\left(r - \frac{1}{4r}\right)\sin\theta \qquad (2.69)$$
$$+ \alpha\left\{ \frac{B_0 u_0(z)}{r} + \sum_{\ell=1}^{\infty} B_\ell u_\ell(z) K_1\left(\frac{k_\ell r}{\sqrt{S}}\right)\right\}\sin\theta + O(\alpha^2)$$

and

$$T^{(0)} = -u_0'(z)\left(r - \frac{1}{4r}\right)\sin\theta \qquad (2.70)$$
$$+ \alpha\left\{ \frac{B_0 u_0'(z)}{r} + \sum_{\ell=1}^{\infty} B_\ell u_\ell'(z) K_1\left(\frac{k_\ell r}{\sqrt{S}}\right)\right\}\sin\theta + O(\alpha^2)$$

The velocity shear at the sides of the island ($\theta = \pm 90$) is:

$$|\Psi_{rz}|_{\theta=90^\circ} = 2\, u_0'(z) + O(\alpha) \qquad (2.71)$$

or exactly twice its value upstream. At $r=\frac{1}{2}$ the temperature perturbation vanishes to $O(\alpha^{\circ})$ making the island surface temperature a function of z only. Such a form of the temperature implies that the fluid moves only horizontally and the result is shown in figure(17a). The isotherms slope downward from left to right facing downstream to support a positive shear (i.e. $b > 0$) and the slope increases near the island where the shear is intensified.

A small deviation from vertical sides causes the velocity of the deeper fluid to be increased relatively more than that of the fluid above. For a and b both greater than zero the shear is decreased from that found for the pure cylinder model and the isotherms are brought more into the upstream form. Figure (17b) illustrates this point while in figure (18) depth contours for an isotherm are drawn. There is little similarity between these pictures and the data of figures (9) to (14). In particular, the depression of the isotherms on the upstream end of the island is not explained.

There is, however, one way of producing such a feature within the concepts of this model. If the current is flowing in the negative x-direction below some depth while the shear is still positive (i.e. $a < 0$, $b > 0$), then the isotherms far from the island will retain the slope of the previous discussion but the shear magnitude will be increased by the side slope of the island in the region of reverse flow. This point is illustrated in figure (19). The region of depressed isotherms and closed contours occurs only in the lower portion of the reversed flow. For the Gosnold 144 data the lowering of the isotherms

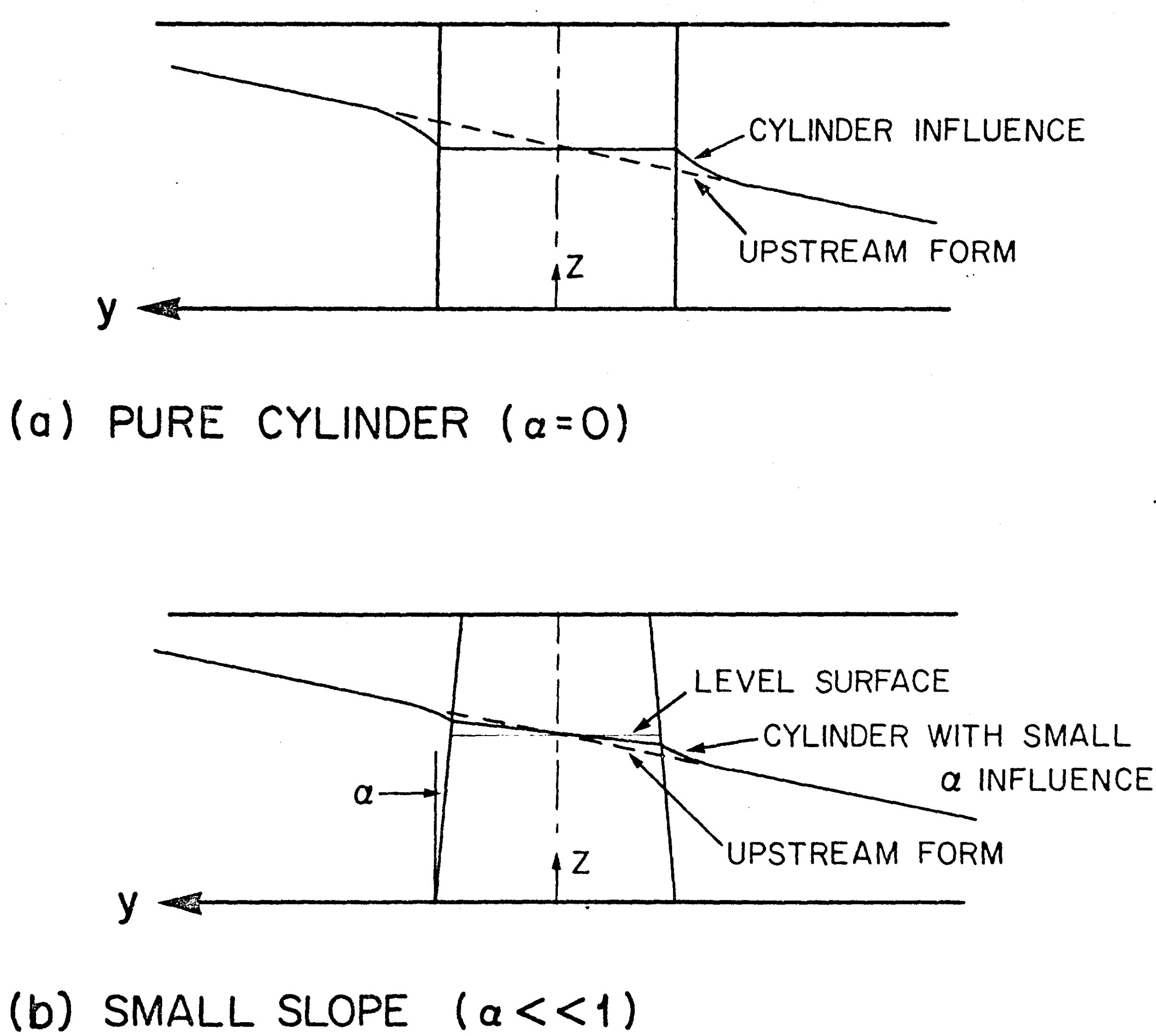
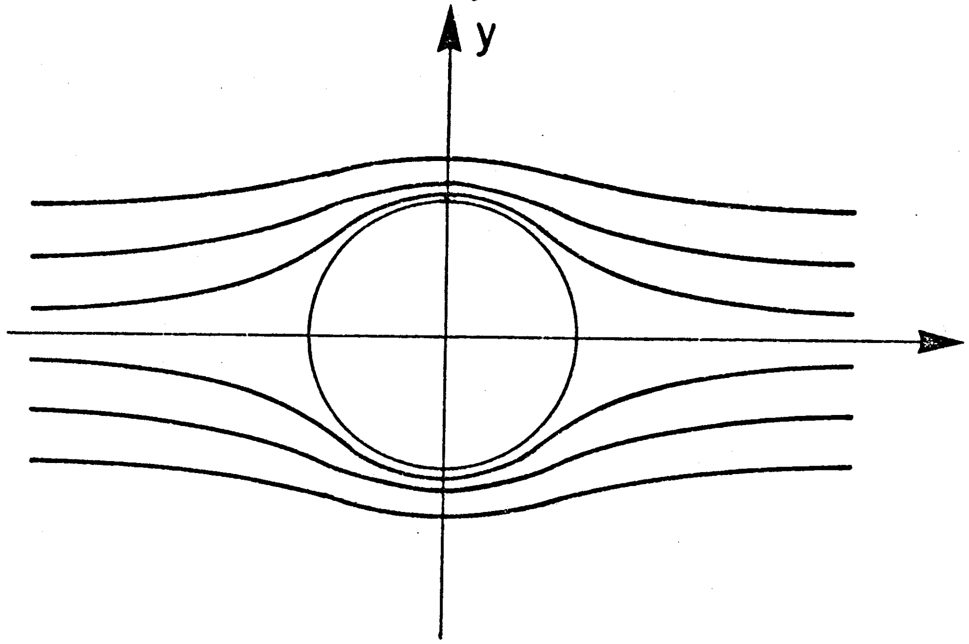
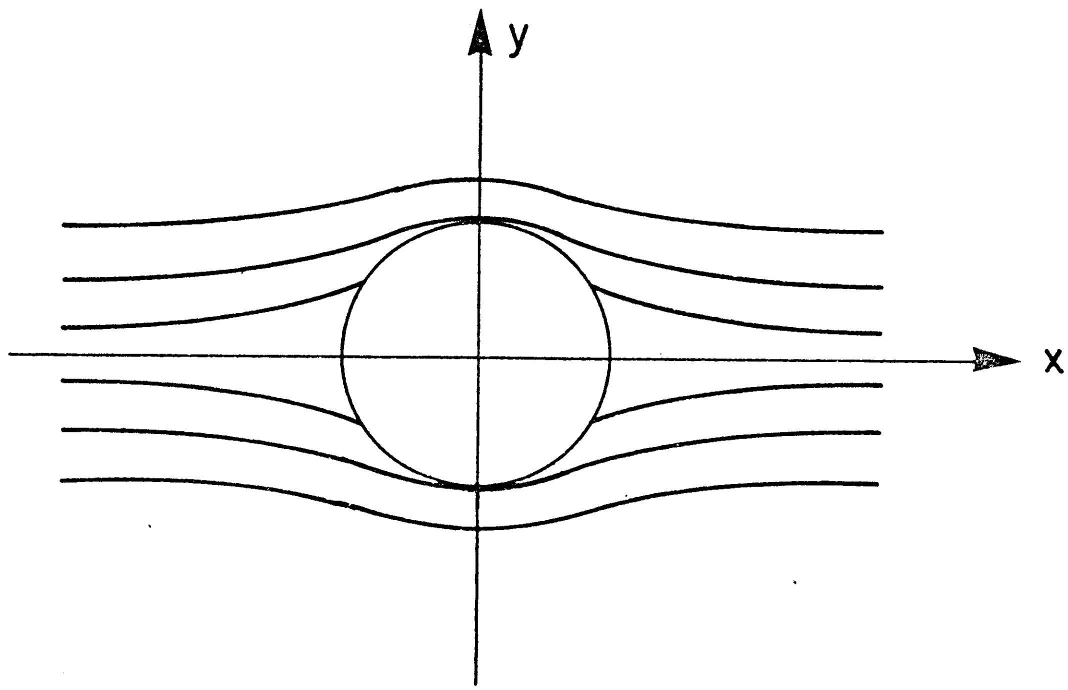



Figure 17. The line of intersection of an isothermal plane with the obstacle for no circulation.




(a) $\alpha = 0$



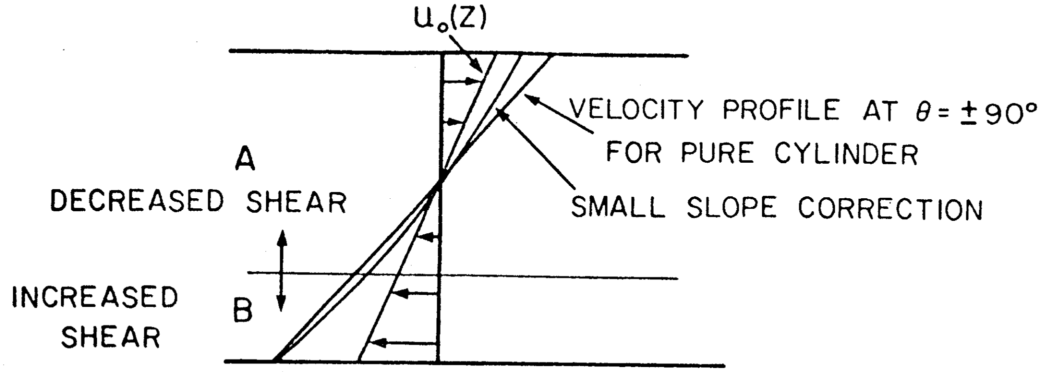

(b) $\alpha << 1$

Figure 18. A rough sketch of isotherm depth contours for $a>o$ and $b>o$ ($u_o(z)=a+bz$).

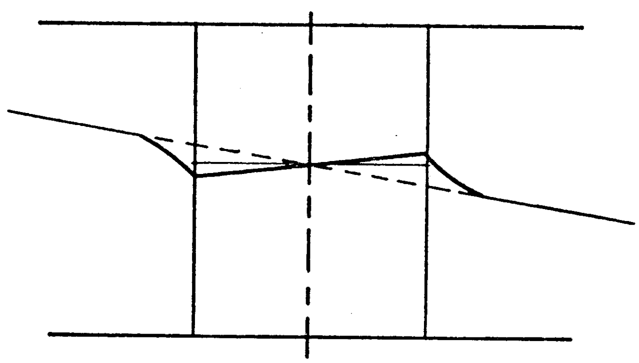
would be $O(\epsilon \alpha bH)$ or about 35m which is not too far from the observed values of 50 to 100m. However, current measurements, although few in number, did not indicate a reversal in current direction with depth, certainly not in the upper 500m where a definite depression is found in the data. In addition the observed region of closed streamlines (according to interpretation A) is not centered at the point of maximum horizontal deflection of the current but closer to the stagnation point.

Such a picture might conform better to the Atlantis II cruise where a reversal of flow was observed between the surface and 600m and the maximum depression and elevation of the isotherms is more towards the sides of the island. For this case $\epsilon = .05$ and the magnitude of the effect is about 17m or about what is observed. In addition $\epsilon \ll \alpha$ and the effects discussed in the next section are smaller than those described here.

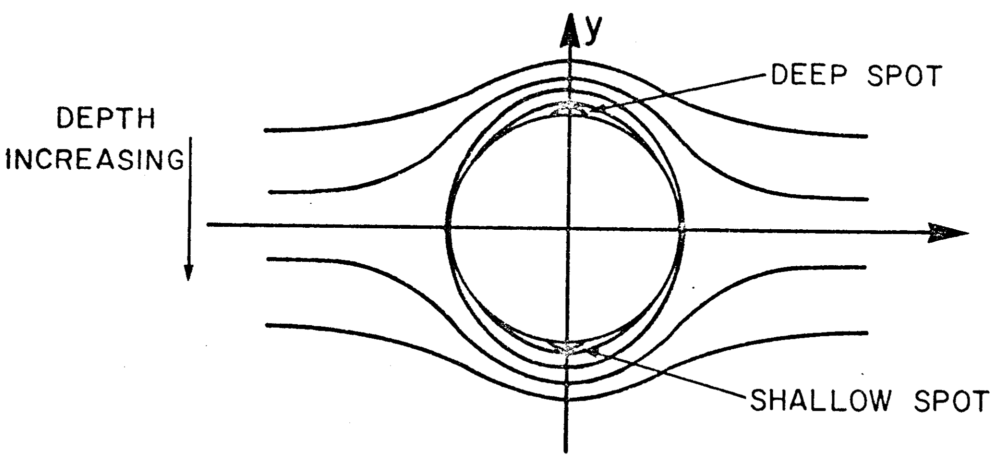
Circulation does not change this conclusion that the slope effect alone is insufficient to produce the required isotherm pattern for Gosnold 144 data. From equation (2.68) the temperature perturbation $T^{(0)}$ is constant at $r = \frac{1}{2}$ to $O(\alpha^{\circ})$. A nonzero circulation displaces the stagnation point off the x-axis and distorts the isotherms enough to balance the altered shear. The slope effects will be similar to those discussed in the zero circulation case and, as the circulation contribution to $T^{(0)}$ in equation (2.68) is independent of $\theta$ it cannot produce the temperature variations that are observed. These are strong functions of angle and are not reproduced by the $O(\alpha)$ slope effects.




(a) VELOCITY PROFILE AT $r=1/2, \theta=\pm 90°$



(b) INTERSECTION OF AN ISOTHERMAL PLANE WITH THE ISLAND




(c) DEPTH CONTOURS OF A SELECTED ISOTHERM IN REGION B

Figure 19. The effect of island slope on an isotherm when $a<o$, $b>o$ ($u_o(z) = a+bz$).

2.4. The O($\epsilon$) Problem (no circulation)

It was concluded in the last section that O($\alpha$) slope effects on the flow could explain data from the Atlantis II cruise but were not sufficient to produce the isotherm contours observed for Gosnold 144. Finite Rossby number effects for the latter cruise, however, are of equal importance with $\epsilon$ =.12 while $\alpha$ =.13. The purpose of this section is to explore the O($\epsilon$) problem in order to see if effects at this level of approximation in the equations can be used to explain the data. In the analysis that follows the circulation function G(z) will be taken to vanish. Unfortunately there is no good physical reason for making this choice within the scope of these arguments. However, the drogue data presented in figure (15) does not indicate the existence of strong circulation, at least at the drogue depths, and the mathematics is considerably simplified with this assumption.

The potential vorticity equation (2.18) expanded to O($\epsilon$) is:

$$\vec{u}^{(0)}\cdot\nabla\left(\zeta^{(1)}+\frac{T_z}{S}^{(1)}\right)+\vec{u}^{(1)}\cdot\nabla\left(\zeta^{(0)}+\frac{T_z}{S}^{(0)}\right) \tag{2.72}$$
$$=\vec{\omega}^{(0)}\cdot\nabla w^{(1)}+\vec{\omega}^{(1)}\cdot\nabla w^{(0)}-\vec{u}_z^{(0)}\cdot\nabla\frac{T}{S}^{(1)}-\vec{u}_z^{(1)}\cdot\nabla\frac{T}{S}^{(0)}$$

From the O($\alpha^0$) problem $\zeta^{(0)}+T_z^{(0)}/S=0$ when $u_0(z)$ is linear and $w^{(0)}=0$ as well. The equations of motion when expanded to O($\epsilon$) are:

$$\vec{u}^{(0)}\cdot\nabla u^{(0)} - v^{(1)} = -p_x^{(1)} \tag{2.73}$$
$$\vec{u}^{(0)}\cdot\nabla v^{(0)} + u^{(1)} = -p_y^{(1)} \tag{2.74}$$
$$0 = -p_z^{(1)} + T^{(1)} \tag{2.75}$$
$$\vec{u}^{(0)}\cdot\nabla T^{(0)} + Sw^{(1)} = 0 \tag{2.76}$$
$$\nabla\cdot\vec{u}^{(1)} = 0 \tag{2.77}$$

The flow remains hydrostatic but there is a correction to the geostrophic balance. With G(z)=0 equation (2.76) gives:

$$\begin{aligned} w^{(1)} &= -\frac{1}{S}\,\vec{u}^{(0)}\cdot\nabla T^{(0)} \\ &= -\frac{1}{S}\,J(\psi,\psi_z) \\ &= 0 \end{aligned} \tag{2.78}$$

because $\psi_z = u_0'(z)\psi/u_0(z)$ to $O(\alpha^0)$ and the Jacobian determinant vanishes. The flow, therefore, remains horizontal at $O(\epsilon)$ and a new streamfunction $\phi(x,y,z)$ can be defined according to:

$$\begin{aligned} u^{(1)} &= -\phi_y \\ v^{(1)} &= \phi_x\,. \end{aligned} \tag{2.79}$$

Equations(2.73) to (2.77) reduce to the following three.

$$-J(\psi,\psi_y) - \phi_x = -p_x^{(1)} \tag{2.80}$$
$$J(\psi,\psi_x) - \phi_y = -p_y^{(1)} \tag{2.81}$$
$$0 = -p_z^{(1)} + T^{(1)} \tag{2.82}$$

Using (2.79) to (2.80) the remaining two terms on the right hand side of (2.72) can be evaluated to give:

$$\vec{u}_z^{(0)} \cdot \nabla \frac{T}{S}^{(1)} + \vec{u}_z^{(1)} \cdot \nabla \frac{T}{S}^{(0)} = -\frac{1}{S}\left\{ \psi_{xz} \frac{\partial}{\partial z} J(\psi, \psi_x) + \psi_{yz} \frac{\partial}{\partial z} J(\psi, \psi_y) \right\}$$

$$= -\frac{2}{S}\left\{ \frac{u_0'(z)}{u_0(z)} \right\}^2 J\left(\psi, \frac{|\nabla \psi|^2}{2}\right) \qquad (2.83)$$

Expressed in terms of pressure, the O($\epsilon$ ) potential vorticity is:

$$\rho^{(1)} + \frac{T_z}{S}^{(1)} = \nabla_h^2 p^{(1)} + \frac{p_{zz}^{(1)}}{S} - 2J(\psi_x, \psi_y) \qquad (2.84)$$

and (2.72) becomes:

$$J\left(\psi, \nabla_h^2 p^{(1)} + \frac{p_{zz}^{(1)}}{S} - 2J(\psi_x, \psi_y) - \frac{2}{S}\left\{\frac{u_0'(z)}{u_0(z)}\right\} \frac{|\nabla_h \psi|^2}{2}\right) = 0 \qquad (2.85)$$

This Jacobian relation can be integrated andthe unknown function of $\psi$ evaluated far from the island's influence where $p^{(1)}=0$ and $\psi=-u_0(z)y$. One finds that:

$$\nabla_h^2 p^{(1)} + \frac{p_{zz}^{(1)}}{S} = 2J(\psi_x, \psi_y) + \frac{2}{S}\left\{\frac{u_0'(z)}{u_0(z)}\right\}\left\{\frac{|\nabla_h \psi|^2 - u_0^2(z)}{2}\right\} \qquad (2.86)$$

Substitution of the O($\alpha^0$) results into this relation gives:

$$\nabla_h^2 p^{(1)} + \frac{p_{zz}^{(1)}}{S} = -\frac{u_0^2(z)}{2r^6} + \frac{u_0'^2}{2Sr^2}\left(\frac{1}{8r^2} - \cos 2\theta\right) \qquad (2.87)$$

which is an inhomogeneous Laplace's equation (i.e. a Poisson's equation).

The boundary condition at the island, which can now be considered a right circular cylinder, is that the radial velocity vanish there meaning that:

$$\phi = \phi(z) \quad \text{on} \quad r = \frac{1}{2}$$

from arguments similar to those used at $O(\epsilon^0)$. Consistent with the assumption of zero circulation this function of z will be taken to vanish and:

$$\phi = 0 \quad \text{on} \quad r = \frac{1}{2}\,. \tag{2.88}$$

If (2.73) is integrated with respect to x and (2.74) with respect to y, the relationship between $p^{(1)}$ and $\phi$ is found to be:

$$p^{(1)} = \phi + \frac{u_0^2(z)\cos 2\theta}{4r^2} - \frac{u_0^2(z)}{32r^4} + \mathcal{F}(z)\,. \tag{2.89}$$

The function entering from the integration, $\mathcal{F}(z)$, must vanish as both $p^{(1)}$ and $\phi$ go to zero as $r \to \infty$. Therefore:

$$p^{(1)} = \frac{u_0^2(z)\cos 2\theta}{4r^2} - \frac{u_0^2(z)}{32r^4} \quad \text{on} \quad r = \frac{1}{2}, \tag{2.90}$$

is the appropriate condition on $p^{(1)}$.

The conditions on z=0 and z=1 are derived from the $O(\epsilon^2)$ heat equation which is:

$$\vec{u}^{(0)}\cdot\nabla T^{(1)} + \vec{u}^{(1)}\cdot\nabla T^{(0)} + S w^{(2)} = 0\,.$$

Putting in the dependences on $\psi$ and $p^{(1)}$ and making $w^{(2)}=0$ one finds that:

$$J\left(\psi,\, p_z^{(1)} - \frac{u_0'(z)}{u_0(z)}\, p^{(1)} - \frac{u_0'(z)}{u_0(z)}\, \frac{|\nabla\psi|^2}{2}\right) = 0 \quad \text{on } z = \binom{0}{1}.$$

This expression can be integrated and the integration function of $\psi$ evaluated in the usual manner to find that:

$$p_z^{(1)} - \frac{u_0'(z)}{u_0(z)}\, p^{(1)} = \frac{u_0'(z)}{u_0(z)} \left\{ \frac{|\nabla\psi|^2 - u_0^2(z)}{2} \right\}$$

$$= \frac{u_0(z)\,u_0'(z)}{4r^2} \left\{ \frac{1}{8r^2} - \cos 2\theta \right\} \text{ on } z = \binom{0}{1}. \quad (2.91)$$

The final condition on $p^{(1)}$ is:

$$p^{(1)}(r,z,\theta) \to 0 \text{ as } r \to \infty. \qquad (2.92)$$

Before proceeding further with the analysis an idea of the nature of the result can be obtained from the boundary condition at the cylinder wall. Differentiating (2.90) with respect to z yields:

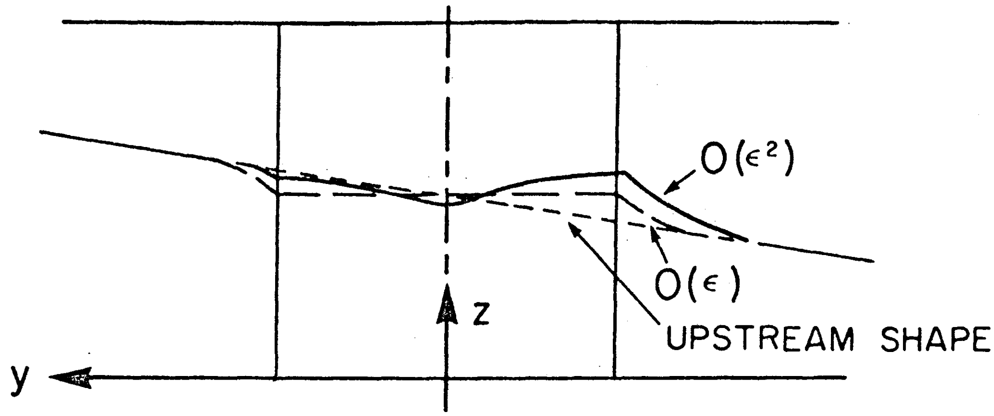
$$T^{(1)}(r=\tfrac{1}{2},\theta,z) = p_z^{(1)}(r=\tfrac{1}{2},\theta,z)$$

$$= 2u_0(z)\,u_0'(z)\left(\cos 2\theta - \tfrac{1}{2}\right) \qquad (2.93)$$

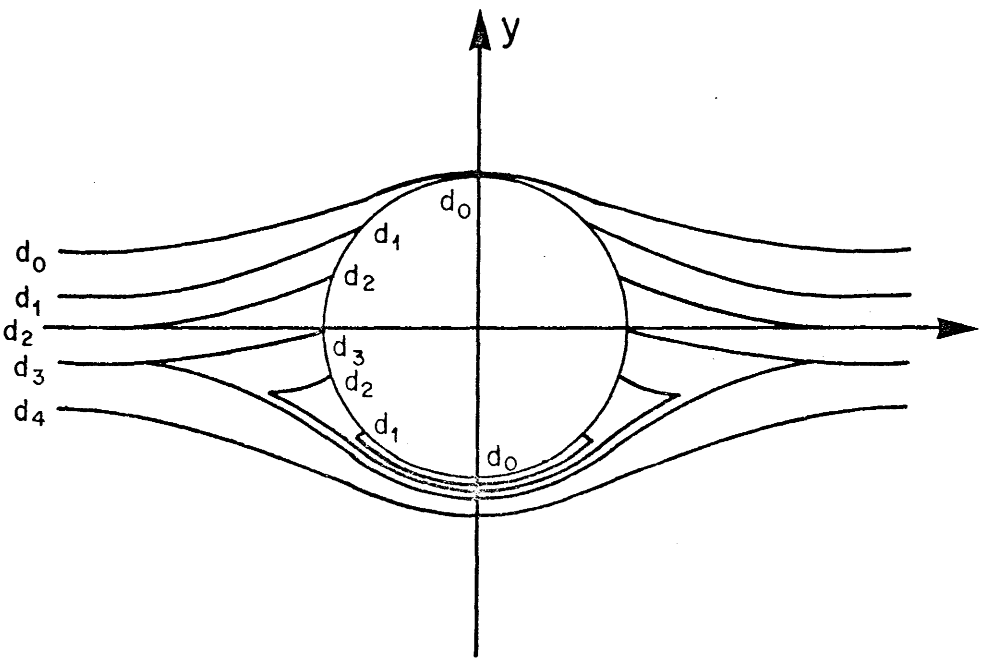
There is a zero perturbation temperature at $O(\epsilon)$ for $\theta = 30°$, 150°, 210°, and 330°, and the shape of an isothermal plane along the line of intersection with the cylinder will be something like the situation depicted in figure (20 a). Note the relatively narrow (60° wide) depression around the stagnation point. The depth of this depression over the depth at $\theta = \pm 90°$ is found by substituting (2.83) into the temperature scaling equation (2.5) and evaluating the difference in z for T constant between $\theta = 90°$ and $\theta = 180°$. This is:

$$\Delta z = \frac{4\epsilon^2}{S}\, u_0(z)\, u_0'(z). \qquad (2.94)$$

If the scaled velocity and shear are both about unity, the




(a) INTERSECTION OF AN ISOTHERMAL PLANE WITH THE CYLINDRICAL ISLAND WITH O($\epsilon^2$) CORRECTION




(b) ROUGH SKETCH OF EXPECTED DEPTH CONTOURS FOR AN ISOTHERM

Figure 20. The finite Rossby number influence on the isotherms.

depth of the depression is approximately 80m for Gosnold 144, a value which agrees quite well with the observations of figures (7) to (14). Introduction of slope to the island sides will leave the isotherm unchanged at the stagnation point and move it up on the left and down on the right, as is obseved. Section (2.3) gives these $O(\epsilon\alpha)$ displacements which can be superimposed on the effects described above. Joining the predicted temperature distribution at the island with the far field in an asymptotic manner gives an overall isotherm contour picture like that sketched in figure (20b). The result is similar to interpretation B of the data.

Finding the exact, theoretical form of the isotherms is a somewhat involved problem in partial differential equations. Equation (2.87) is an inhomogeneous Laplace's equation which must be soved in a three-dimensional region exterior to a cylinder with boundary conditions which are both mixed and inhomogeneous.

The analysis proceeds in a simpler fashion if the following substitution is made.

$$p^{(1)}(r,\theta,z) = P(r,\theta,z) - \frac{u_o^2(z)}{4r^2}\cos 2\theta + \frac{u_o^2(z)}{32r^4} \qquad (2.95)$$

Equation (2.87) and the conditions (2.90), (2.91), and (2.92) are transformed into:

$$\nabla_h^2 P + \frac{P_{zz}}{S} = -\frac{u_o^2(z)}{r^6} \qquad (2.96)$$

with

$$P = 2u_o^2(z)\cos 2\theta - u_o^2(z) \text{ on } r = \tfrac{1}{2}, \qquad (2.97)$$

$$P \rightsquigarrow 0 \quad \text{as } r \rightsquigarrow \infty , \tag{2.98}$$

and

$$P_z - \frac{u_0'(z)}{u_0(z)} P = 0 \quad \text{on } z = \binom{0}{1} . \tag{2.99}$$

The angular dependence of $P(r,z,\Theta)$ can be removed from the problem through the substitution:

$$P(r,\theta,z) = P_0(r,z) + P_2(r,z)\cos 2\theta \tag{2.100}$$

with the result that two, two-dimensional problems are created. The $P_2(r,z)$ analysis is the simpler. Here:

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} P_2 - \frac{4}{r^2} P_2 + \frac{\partial^2}{\partial z^2} P_2 = 0 \tag{2.101}$$

with

$$P_2 = 2\, u_0^2(z) \qquad \text{on } r = \tfrac{1}{2} , \tag{2.102}$$

$$P_2 \rightsquigarrow 0 \qquad \text{as } r \rightsquigarrow \infty , \tag{2.103}$$

and

$$P_{2z} - \frac{u_0'(z)}{u_0(z)} P_2 = 0 \qquad \text{on } z = \binom{0}{1} . \tag{2.104}$$

The solution is easily found to be

$$P_2(r,z) = \sum_{n=1}^{\infty} A_n u_n(z) K_2\left(\frac{k_n r}{\sqrt{S}}\right) + \frac{A_0 u_0(z)}{r^2} \tag{2.105}$$

where $u_n(z)$ and $k_n$ are defined by equations (2.54) and (2.55) and:

$$A_0 = \frac{1}{2} \frac{\int_0^1 u_0^3(z)\,dz}{\int_0^1 u_0^2(z)\,dz} , \tag{2.106}$$

$$A_n = 2 \frac{\int_0^1 u_0^2(z)\, u_n(z)\,dz}{K_2\left(\frac{k_n}{2\sqrt{S}}\right)\int_0^1 u_n^2(z)\,dz} . \tag{2.107}$$

$P_0(r,z)$ is somewhat more difficult to obtain. The equation is:

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} P_o + \frac{\partial^2}{\partial z^2} P_o = -\frac{u_o^2}{r^6} \tag{2.108}$$

with boundary conditions:

$$P_o = -u_o^2(z) \qquad \text{on} \quad r = \tfrac{1}{2}\,, \tag{2.109}$$

$$P_o \rightsquigarrow 0 \qquad \text{as} \quad r \rightsquigarrow \infty\,, \tag{2.110}$$

and

$$P_{oz} - \frac{u_o'(z)}{u_o(z)} P_o = 0 \qquad \text{on} \quad z = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{2.111}$$

This is the description of a boundary value problem for an inhomogeneous elliptic equation in a region exterior to a vertical cylinder and bounded by two horizontal planes. Use can be made of Weber's Integral Theorem (see Duff and Naylor) to transform the radial derivatives leaving a one-dimensional problem which can be easily solved. However, the inversion of this transform is rather tedious and, in the end, the result can only be expressed in an integral fashion which then has to be evaluated numerically. In existence on the M.I.T. time sharing system is "EPS - an interactive system for solving elliptic boundary-value problems" (Tillman (1969)). In place of the analytic treatment, this program was used to solve the problem for $P_o(r,z)$ using values of a, b, and S appropriate to the Gosnold 144 cruise.

## 2.5. Discussion

For comparison with the data, the quantity of most interest is the depth of an isotherm as a function of horizontal position. The temperature field is given by the scaling formula in (2.5) where $T'(x',y',z')$ has been calculated in sections (2.4) and (2.5) to $O(\alpha)$ and $O(\epsilon)$ respectively. Because the displacement of an isothermal surface from a smooth horizontal plane is, at most, $O(\epsilon)$ near the island, the height $z'(r',\theta;\alpha,\epsilon)$ of the isotherm above the bottom can also be expanded in a Rossby number power series according to:

$$z'(r',\theta;\alpha,\epsilon) = z_0'(r',\theta;\alpha) + \epsilon z_1'(r',\theta;\alpha) + \\ + \epsilon^2 z_2'(r',\theta;\alpha) + O(\epsilon^3) \tag{2.112}$$

Substituting this into the temperature scaling, and then using a Taylor series expansion and the fact that $T_{z'}'^{(0)}=0$ to $O(\alpha^0)$ one finds that:

$$z_0' = \frac{T - T_0}{\Delta T}, \tag{2.113}$$

$$z_1' = -\frac{1}{S} T^{(0)}(x', y', z_0'; \alpha) \tag{2.114}$$

and

$$z_2' = -\frac{1}{S} T^{(1)}(x', y', z_0') \tag{2.115}$$

Letting $\Delta z'$ be the scaled departure from a level surface at height $z_0'$, and writing the temperature perturbations $T^{(0)}$ and $T^{(1)}$ in terms of $\psi = \psi^{(0)} + \alpha\psi^{(1)}$ and $p^{(1)}$ respectively, then:

$$\Delta z' = -\frac{\epsilon}{S}\psi^{(0)} - \frac{\epsilon}{S}\left[\alpha \psi_z^{(1)} + \epsilon p_z^{(1)}\right] \tag{2.116}$$

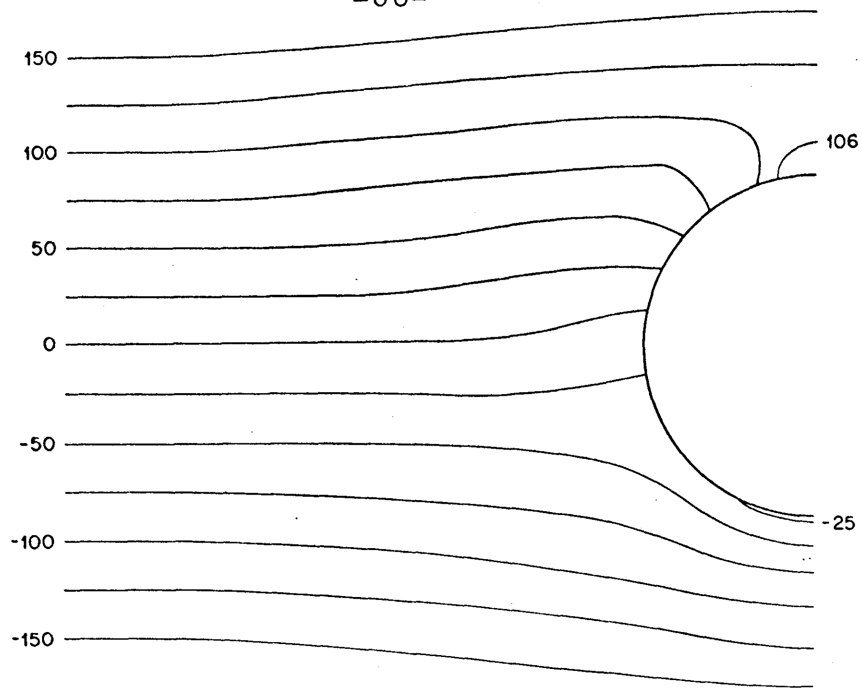
$\psi^{(0)}$ and $\psi^{(1)}$ are expressed analytically in series form by (2.45), (2.50), (2.53) and (2.63) when the circulation function G(z)=0. These infinite series were evaluated by a numerical summation that was terminated when errors were sufficiently small. The $O(\epsilon)$ pressure $p^{(1)}$ is determined from (2.95) and (2.100) and the resulting two problems in $P_o$ and $P_2$. $P_2$ is given analytically by the series expression (2.105) while $P_o$ was calculated by the numerical integration scheme in existence on the M.I.T. computer time sharing system. In the computations the following values for the parameters were chosen.

$$\begin{aligned} a &= 0 \\ b &= 1.0 \\ S &= 1.0 \\ \epsilon &= .12 \\ \alpha &= .13 \end{aligned} \qquad (2.117)$$

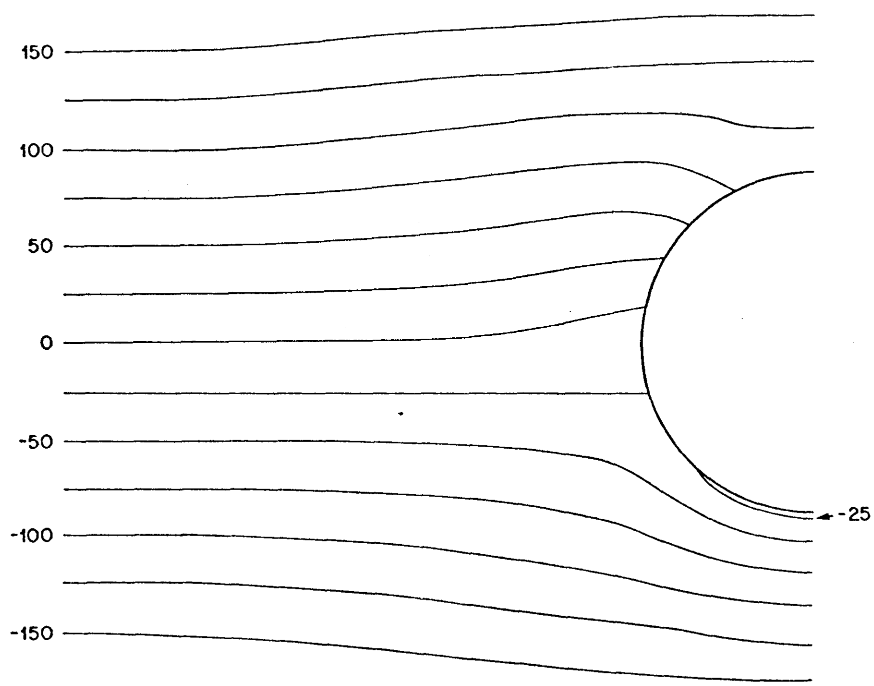
These are representative of the conditions found during the later part of Gosnold 144. With a bottom assumed at 1500m the four isotherms presented in figures (7) to (14) are approximately at the scaled heights of $z_o'=.4$ (6°C), $z_o'=.5$ (10°C), $z_o'=.6$ (14°C), and $z_o'=.7$ (17°C).

The isotherm displacement, $\Delta z'$, was evaluated at each of the four levels on a polar grid in which the radial spacing was inversely proportional to the distance from the origin. Values were plotted by a Calcomp routine and then isotherm depth contours were interpolated by hand at depths corresponding to 25m increments. The results are shown in figure (21).

The resemblance between the picture presented by this

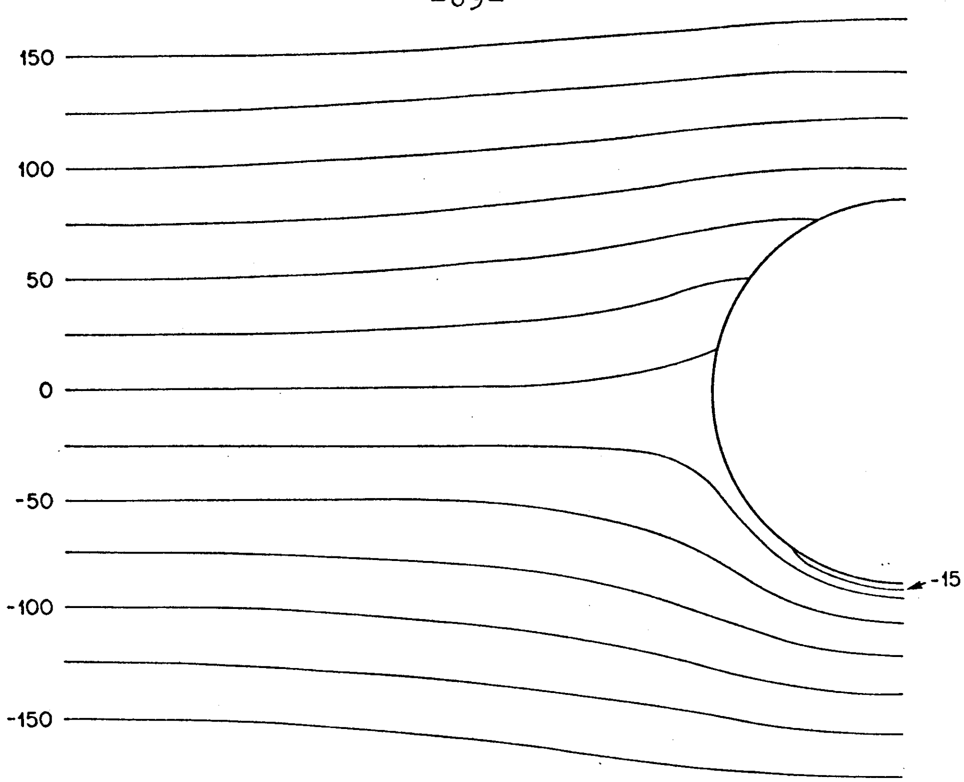



(a) $Z_o' = .7$ ; 17° C.

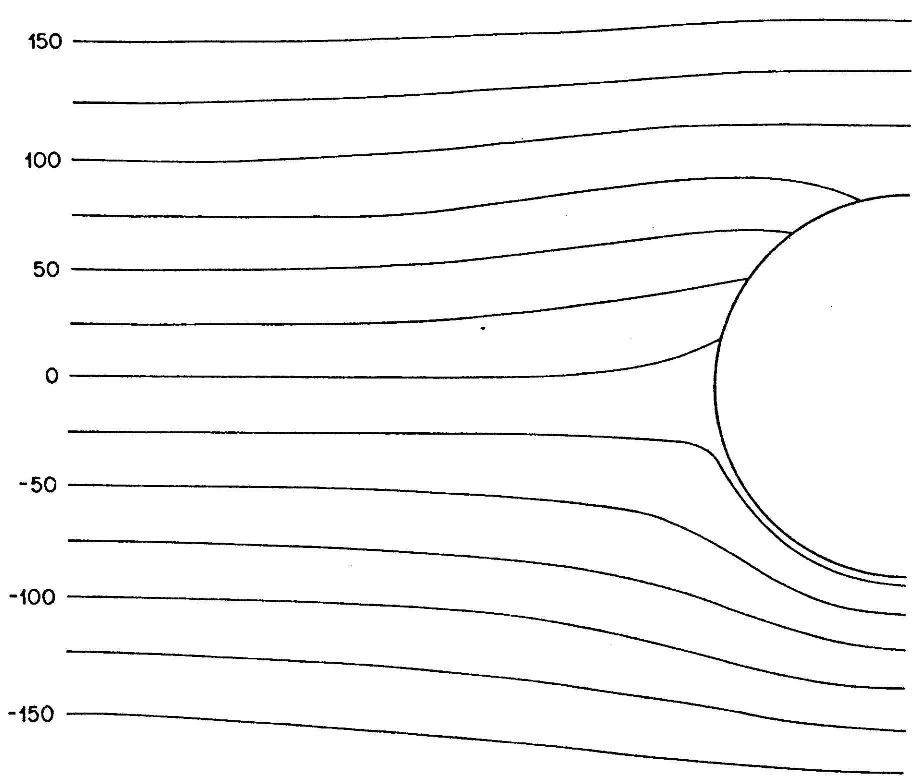



(b) $Z_o' = .6$ ; 14° C.

Figure 21. The computed isotherm depth contours in meters from the average position with finite Rossby number and slope influences when $\varepsilon = .12$, $\alpha = .13$, and $S = 1.0$ for (a) $z' = .7$, (b) $z' = .6$, (c) $z' = .5$, and (d) $z' = .4$.




(c) $Z_o' = .5$ ; 10° C.




(d) $Z_o' = .4$ ; 6° C.

analysis and the isotherm contours of interpretation B is striking. In both cases the contours intersect the island boundary on the north and are deflected away from it in the south by a closed elevation of the isotherms. The height of this hill is not well reproduced but better agreement could be achieved by a different choice of the bottom depth. Slope and Rossby number effects combine on the north boundary to raise the isotherms while to the south the slope influence detracts from the rise produced by the $O(\epsilon)$ problem. A shallower depth reduces $\alpha$ and increases $\epsilon$ and therefore increases the height of this elevation. In the model a linear upstream velocity profile was used and in applying this model to Bermuda a trade off must be made between the real bottom depth and the depth at which the current becomes small. 1.5km was thought to be a good value but this number can be adjusted to give better agreement.

However, it is more likely that the lack of quantitative agreement is a result of the violation of other assumptions. The flow does not appear to be either steady or uniform upstream from the island. Unfortunately stations were not taken far enough from the island's influence to obtain a good picture of the undisturbed flow but the current did undergo a change in direction and magnitude during the cruise. The contours of the data in figures (7) to (14) do not show a uniform flow at their furthest extent from the island but, instead, the closely spaced contours to the west best represent a narrow intense current. Whether this is an island effect or not is difficult to tell. Another complication comes from the geometry. Bermuda is not the isolated circular cone that was used in the model but it has a definite ellipticity and possesses two banks to the southwest which almost rise to the surface. These features will certainly change the idealized flow pattern to the south. Finally, the neglected circulation, if it exists, will alter the picture presented by this theory.

In spite of these far from ideal conditions, the agreement between fact and theory is remarkabley good. Note that even the small predicted reversal in the 17°C isotherm to form another small closed hill on the north shore is found in the data. This isotherm rises from a depth of 342m at station 62 to a depth of 334m at station 63 near the island shore. In the lee of the island the deepest isotherm position usually occurs at the furthest eastern point as expected but agreement is not generally as good as in the forward part of the flow. Either this is because the streamlines detach from the boundary to form a wake (see section (2.6)) or, as was commented on in the data introduction, these stations were taken during a different flow regime. It is also possible that the banks distort this part of the flow beyond recognition.

The other interesting feature noted in the data was the existence of an isolated area on the northern slope where mixing processes were in operation. One possible explanation is that, as the flow diverges around the island, the shear and temperature gradient are so altered that the flow becomes unstable to small perturbations. A measurement of the stability is given by the local Richardson number $Ri(r,\theta,z)$ which is a ratio of the temperature gradient and the square of the velocity shear and is defined by:

$$Ri(r,\theta,z) = \frac{g\alpha \frac{\partial T}{\partial z}}{\left|\frac{\partial \vec{q}}{\partial z}\right|^2} \qquad (2.118)$$

in dimensional terms where $\vec{q}$ is the horizontal velocity. A useful criterion for stability is that $Ri > 1/4$.

Far from the island the Richardson number is:

$$Ri(r,\theta,z) \cong \frac{g\alpha\,\Delta T/H}{(\partial u_0/\partial z)^2} = \frac{N^2}{(\partial u_0/\partial z)^2}$$

which gives $Ri \approx 88$ for the conditions prevailing in Gosnold 144. The flow is definitely stable under this criterion in this region. Substituting the scaling transformations of (2.5) into the Richardson number definition one finds that:

$$Ri(r,\theta,z) = \left(\frac{HN}{U}\right)^2 \frac{\left(1 + \frac{\epsilon}{S}\frac{\partial T'}{\partial z'}\right)}{\left|\frac{\partial \vec{q}'}{\partial z'}\right|^2} . \qquad (2.119)$$

To lowest order:

$$\left|\frac{\partial \vec{q}'}{\partial z}\right|^2 = \frac{1}{r^2}\left(\psi^{(0)}_{rz}\right)^2 + \left(\psi^{(0)}_{rz}\right)^2 + O(\epsilon,\alpha) \qquad (2.120)$$

$$= u_0'(z)\left(1 + \frac{1}{16r^4} - \frac{1}{2r^2}\cos 2\theta\right) + O(\epsilon,\alpha)$$

which rises to a maximum of $4(u_0'(z))^2$ at $r=1/2$, $\theta = \pm 90°$. Therefore, at the north and south extremes of the island the velocity shear is doubled to O(1) and the Richardson number is reduced by a factor of 4. This effect would reduce the local value of Ri to 22, still far above the critical value. It is possible that there are more localized areas in the water column that have lower stability which is reduced to the critical point but this mechanism still suggests that this occur on both the north and south slopes.

Now consider the higher order slope and Rossby number influences at the two critical points singled out by the O(1) considerations. The expansion for T'(x',y',z') gives:

$$\frac{\partial T'}{\partial z'} = \frac{\partial^2}{\partial z'^2}\left(\psi^{(0)} + \alpha\psi^{(1)} + \epsilon p^{(1)}\right) + O(\epsilon^2, \epsilon\alpha, \alpha^2) \qquad (2.121)$$

From the boundary conditions on $\psi^{(0)}$, $\psi^{(1)}$ and $p^{(1)}$ at r=1/2 which are given by (2.51) (G(z)=0), (2.61) and (2.90):

$$\begin{aligned}\frac{\partial T'}{\partial z} &= -4\alpha - 3\epsilon \quad \text{at } r=\tfrac{1}{2},\ \theta = 90^\circ, \\ &= +4\alpha - 3\epsilon \quad \text{at } r=\tfrac{1}{2},\ \theta = -90^\circ.\end{aligned} \qquad (2.122)$$

and the effect of the island is to decrease the temperature gradient more on the north than on the south. For the values of $\alpha$ and $\epsilon$ used here the gradient is actually increased slightly on the south. This contribution to the Richardson number is, therefore, to make the north slope the more favored area for instabilities to grow.

The higher order effects in the velocity shear at these critical points are more significant as they are O($\epsilon$, $\alpha$) instead of the O($\epsilon^2$, $\epsilon\alpha$) changes to the temperature gradient derived above. However, the streamline pattern has been chosen to be symmetric and this choice forces $|q'_{z'}|$ to be the same on either side of the island. At the island wall the vertical derivative of the azimuthal velocity v' gives the necessary shear. Expanded into its dependence on $\psi$, $\phi$ and p this is:

$$v'_{z'}\Big|_{r=\frac{1}{2}} = \frac{\partial}{\partial z'}\left(\psi_{r'} + \epsilon\,\phi_{r'}\right)\Big|_{r=\frac{1}{2}}$$
$$= \frac{\partial}{\partial r'}\left(\alpha\psi^{(1)}_{z'} + \epsilon\, p^{(1)}_{z'}\right)\Big|_{r=\frac{1}{2}} - 16\epsilon z'\sin^2\theta \qquad (2.123)$$

Unfortunately the series expression for $\psi^{(1)}$ given by (2.63) converges as $k_n^{-2}$ at r=1/2 and the series for $\psi^{(1)}_{rz}$ will, therefore, not converge at the island side. As was discussed in section (2.3) the effect of slope is to reduce the shear and it can be shown from the numerical computations that the nonlinearities also have this tendency. Therefore the stability of the fluid will be increased by these considerations. There will be a higher order contribution of $O(\epsilon^2, \epsilon\alpha, \alpha^2)$ from the velocity shear that competes with the first temperature gradient effect but this cannot be evaluated from the present analysis.

All that can be said is that, as long as circulation is nonexistent, the velocity and velocity shear will be symmetric with respect to the island and the only asymmetry comes from the temperature gradient making the north slope the preferred area for instabilities to occur. Another possible cause of this mixing is presented in section (2.6) which shows that the north slope is the preferred area for streamline separation to form a wake.

## 2.6. Viscous Effects

One of the most tenuous assumptions made in the analysis of this chapter has been that all viscous effects are confined to narrow regions at the solid boundaries and that the interior can be treated inviscidly. This neglect of viscosity gives rise to an indeterminant circulation similar to that of inviscid, homogeneous, nonrotating flows. One might anticipate that other problems which appear in nonrotating flows and are related to this zero viscosity limit might manifest themselves here as well. The phenomenon of streamline detachment and wake formation behind bluff obstacles in high Reynolds number flow conditions is particularily troublesome for its occurance invalidates any solutions based on inviscid conditions applied at rigid boundaries.

Separation of streamlines from the boundary occurs in homogeneous, nonrotating flows as the fluid passes behind the obstacle. Fluid particles are accelerated as they move from the forward stagnation point to the side where the velocity reaches a maximum. From Bernoulli law considerations this results in a pressure minimum at the side and the fluid encounters a pressure hill in the lee of the obstacle. This adverse pressure gradient is transferred into the viscous boundary layer where the slower moving particles do not have the momentum to climb the pressure hill. Instead the streamlines detach from the boundary and the particles leave the rigid boundary near the side. In mathematical terms the adverse pressure gradient shows up as a forcing term in the boundary layer momentum equations. When the streamlines separate a wake is formed which may or may not be steady. This, at least, is the accepted picture for homogeneous, nonrotating flows at high Reynolds number. (see Batchelor chapter 5)

Rotation alters these ideas somewhat. To lowest order in the Rossby number, the work of section (2.3) has shown that the streamfunction and pressure perturbation function are identical to $O(\epsilon^{o})$. As the flow at $O(1)$ is horizontal the pressure felt by the fluid as it goes around the obstacle close to the boundary is constant to $O(1)$. The only pressure gradient is the one normal to the island which balances the Coriolis force.

When transformed into polar coordinates with radial and zonal velocities represented by u and v respectively, equations (2.6) to (2.10) become: ($\beta'=0$)

$$\epsilon\left(u\frac{\partial}{\partial r}+\frac{v}{r}\frac{\partial}{\partial\theta}+w\frac{\partial}{\partial z}\right)u-\epsilon\frac{v^2}{r}-v=-p_r+E\left[\frac{\partial^2 u}{\partial z^2}+\delta^2\left(\nabla_h^2 u-\frac{u}{r^2}-\frac{2}{r^2}\frac{\partial v}{\partial\theta}\right)\right] \quad (2.124)$$

$$\epsilon\left(u\frac{\partial}{\partial r}+\frac{v}{r}\frac{\partial}{\partial\theta}+w\frac{\partial}{\partial z}\right)v+\epsilon\frac{uv}{r}+u=-\frac{1}{r}p_\theta+E\left[\frac{\partial^2 v}{\partial z^2}+\delta^2\left(\nabla_h^2 v-\frac{v}{r^2}+\frac{2}{r^2}\frac{\partial u}{\partial\theta}\right)\right] \quad (2.125)$$

$$\delta^2\epsilon\left(u\frac{\partial}{\partial r}+\frac{v}{r}\frac{\partial}{\partial\theta}+w\frac{\partial}{\partial z}\right)w=-p_z+T+\delta^2E\left[\frac{\partial^2 w}{\partial z^2}+\delta^2\nabla_h^2 w\right] \quad (2.126)$$

$$\epsilon\left(u\frac{\partial}{\partial r}+\frac{v}{r}\frac{\partial}{\partial\theta}+w\frac{\partial}{\partial z}\right)T+Sw=\frac{E}{\sigma}\left[\frac{\partial^2 T}{\partial z^2}+\delta^2\nabla_h^2 T\right] \quad (2.127)$$

$$\frac{1}{r}\frac{\partial}{\partial r}ru+\frac{1}{r}\frac{\partial v}{\partial\theta}+\frac{\partial w}{\partial z}=0 \quad (2.128)$$

In the viscous boundary layer v must be $O(1)$ as it is brought from an $O(1)$ value to zero. The heat equation shows that w is at most $O(\epsilon)$ and therefore u is $O(\Delta)$ from the continuity equation if $\Delta$ is the boundary layer thickness. These scalings make the nonlinear and viscous terms in the radial momentum balance small with respect to unity. The only term left to balance the Coriolis force is the radial pressure gradient meaning that the correction to p is $O(\Delta)$. The vertical balance shows that the temperature perturbation is $O(\Delta)$.

Defining a scaled radial coordinate $\eta$ by:

$$\eta = \frac{r - \frac{1}{2}}{\Delta} \tag{2.129}$$

and boundary layer variables by:

$$\begin{aligned} (u,v,w) &= (\Delta\tilde{u}, \tilde{v}, \epsilon\tilde{w}) \\ p &= \tilde{p} \\ \text{and} \quad T &= \tilde{T}\ , \end{aligned} \tag{2.130}$$

the lowest order balance in the boundary layer is:

$$-\tilde{v} = -\tilde{p}_\eta \tag{2.131}$$

$$\epsilon\left(\tilde{u}\frac{\partial\tilde{v}}{\partial\eta} + 2\tilde{v}\frac{\partial\tilde{v}}{\partial\theta}\right) = \frac{\delta^2 E}{\Delta^2}\ \frac{\partial^2\tilde{v}}{\partial\eta^2} \tag{2.132}$$

$$0 = -\tilde{p}_z + \tilde{T} \tag{2.133}$$

$$\tilde{w} = 0 \tag{2.134}$$

$$\frac{\partial\tilde{u}}{\partial\eta} + 2\frac{\partial\tilde{v}}{\partial\theta} = 0\ . \tag{2.135}$$

From the azimuthal momentum balance the length parameter $\Delta$ can be found to be:

$$\begin{aligned} \Delta &= \delta\left(\frac{E}{\epsilon}\right)^{1/2} \\ &= \frac{H}{L}\cdot\left(\frac{fL}{U}\cdot\frac{\nu}{fH}\right)^{1/2} \\ &= \left(\frac{1}{Re}\right)^{1/2}. \end{aligned} \tag{2.136}$$

Equations (2.132) and (2.135) form a closed set for $\tilde{u}$ and $\tilde{v}$

which is identical to the set of equations describing uniform, homogeneous flow over a flat plate at high Reynolds number. In the latter problem when the flow is nonuniform an additional term is present in the equation corresponding to (2.132) which is the pressure gradient along the plate in the interior flow. The term is absent in the rotating problem because there is no such interior pressure gradient.

No solutions to this problem have been found but the behavior might be expected to resemble that of nonrotating flow over a flat plate. There, the actual boundary layer thickness grows as $\left(\frac{\nu x}{U}\right)^{1/2}$ where x is the downstream position from the leading edge of the plate. In the rotating case the behavior could be proportional to $\left(\frac{\theta}{\sin\theta}\right)^{1/2}$ as the interior velocity along the boundary varies as $\sin\theta$. The growth would be slow from the forward stagnation point to the side and rapid from there to the rear stagnation point. However, it has not been possible to obtain similarity solutions of this form.

The conclusion is that for small Rossby number but high Reynolds number separation of the streamlines might not occur in the rotating flow problem because there is no adverse pressure gradient along the boundary. Note, however, that the $O(\epsilon)$ will change the picture. Equation (2.90) for $p^{(1)}$ shows that there is a pressure minimum at the side of the obstacle. To find the pressure forces felt by the fluid the vertical motion must also be taken into account. The downward motion at the forward stagnation point and the upward motion at the sides from the $O(\epsilon)$ influences will reinforce this minimum hydrostatically. Slope effects will increase this minimum on the left and decrease it on the right, once again making the north slope the most likely area for mixing which this time is caused by wake formation.

Boyer (1970) has performed a series of experiments with steady rotating flow past a cylinder and arrived at similar conclusions with regards to the streamline separation phenomenon. His experimental results show that as the Rossby number is increased the flow evolves from an ideal inviscid pattern with fore-aft symmetry to one in which one or more eddies form behind the obstacle and are enclosed within attached streamlines, and finally to complete streamline separation and wake formation. In the experimental situations the Rossby number was still less than unity when this last step occurred but the Reynolds number was as high as $10^3$. The flow of the island problem is stratified as well as rotating and possesses shear but the boundary layer equations for the purely rotating case are identical to those of (2.132) to (2.135).

Therefore, the inviscid linit used in this chapter is quite possibly valid and no streamline detachment or wake formation occurs when the Rossby number is small even though the Reynolds number is much larger than unity.

# 3. SHOALING INTERNAL WAVES

## 3.1. Introduction

There is another possible explanation for the isotherm distortion and mixing phenomenon which cannot be excluded a priori, and this is the nonlinear rectification of shoaling internal waves to produce mean motions. Longuet-Higgins and Stewart (1964) first discussed this mechanism for surface waves and interpreted the nonlinear terms in the averaged equations of motion as a "radiation stress". For waves incident on a beach this stress was supported by a mean pressure gradient produced, in turn, by a slope or "set-down" of the sea surface. In the region between the breaker zone and the shore these results were extended by making suitable assumptions about the amplitude behavior of the motion and then employing the same radiation stress arguments. Agreement between fact and theory is good. The goal of this chapter is to investigate the problem for internal gravity waves incident on a beach to see whether an analogous isotherm tilt is produced.

The understanding of shoaling internal waves is somewhat less than that of surface waves. Wunsch (1968,1969) found exact linear solutions for internal waves in a two-dimensional, semi-infinite wedge. Keller and Mow (1969) using WKB methods studied internal wave propagation in a fluid in which the bottom variations were over a much greater length scale than a wavelength. Their results can be applied to three dimensional problems but the analysis is again linear. Bretherton (1968) developed a generalized WKB formulation for propagation in slowly varying waveguides.

In Bretherton (1969) he discussed the mean effects of plane internal waves propogating in an infinite medium using the concept of a wave packet and treating the parameter representing the size of the nonlinearities as being of the same order as the one indicating the length of the individual waves to the length of the wave packet. Garrett(1968) studied the propagation of internal waves, also in an infinite medium, which possessed a small shear and was able to extend the concept of radiation stress to this case.

Wunsch (1971), using his wedge modes, calculated the mean displacement of the isopycnals produced by the shoaling wave motion through small nonlinearities in the equations of motion. Hewas able to show that this effect could be significant for typical oceanic conditions. The analysis of this chapter confirms this conclusion. For normal incidence and assuming that the bottom changes little over a wavelength, a generalised WKB expansion is used which includes harmonics in a manner similar to the one used by Chu and Mei (1970) in their study of surface waves incident on a beach. This procedure allows one to calculate information concerning the amplitude and wavenumber changes and harmonic generation as well as the mean effects. Bottom variations are shown to have important consequences on the mean quantities even though a slope parameter does not appear in the final result. This is similar to the influence of rotation on mean Lagrangian currents found by Hasselmann (1970). The nonlinear effects on the fundamental wave and harmonics are identical to those found by Thorpe (1969) in his study of nonlinear effects in a uniform waveguide. The purely bottom slope terms are the same as those found by Keller and Mow.

For oblique motion the approach is less general. WKB solutions similar to those of Keller and Mow are derived to order in the bottom slope parameter and these results are then used in the averaged equations including the nonlinear terms to find the mean motions. Relaxation of the normal incidence constraint is shown to have little effect on the magnitude of the expected isopycnal shift. These results are then used to study the possibilities of generating longshore currents inside a breaker zone. The method used is similar to that of Bowen (1969) and Longuet-Higgins (1970) in the analogous surface wave problem. Significant currents are a possibility.

## 3.2. Formulation

The situation now envisioned has a train of internal waves incident on a beach that slopes so gently that a negligible amount of the incoming energy is reflected. Away from the beach, where the bottom is taken to be flat, the amplitude and phase are assumed constant and the wave crests parallel. The motion has been in progress long enough for a quasi-steady regime to develop; quasi-steady implying that the wave amplitude and phase as well as the mean properties of the fluid are unchanging functions of position on the beach.

Choosing coordinates $(x,y,z)$ with respect to the unit triad $\hat{i}$, $\hat{j}$, and $\hat{k}$ where $\hat{k}$ is vertical, $\hat{i}$ is up the beach and $\hat{j}$ is the mutually perpendicular vector in a right handed sense, then the velocity $\vec{u}(x,y,z,t)$, pressure $p(x,y,z,t)$, and density $\rho(x,y,z,t)$ are all functions of these coordinates as well as time $t$. Letting $f$ be the local Coriolis parameter and $-g\hat{k}$ the gravitational force, the equations of motion for an inviscid, incompressible fluid in the f-plane approximation are:

$$\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u} + f\hat{k}\times\vec{u} = -\frac{\nabla p}{\rho} - g\hat{k} \tag{3.1}$$

$$\frac{\partial \rho}{\partial t} + \vec{u}\cdot\nabla\rho = 0 \tag{3.2}$$

$$\nabla\cdot\vec{u} = 0 \tag{3.3}$$

Although not a good island model, the beach bottom will be chosen to be independent of the y coordinate with the form:

$$\begin{aligned} z &= -h_0, \quad x < 0, \\ z &= -h(x), \quad x > 0. \end{aligned} \tag{3.4}$$

The origin of the coordinate system has been positioned at the surface and at the point where the bottom begins to slope (see fig. 22). Longshore bottom changes could be treated in principle but the algebra becomes very involved. The aim in this presentation is to show that significant effects can be produced by the nonlinear terms and a plane beach suits this purpose. If longshore changes are smaller in comparison with the principal bottom slope then these results would be the lowest order in an additional perturbation scheme.

The inviscid boundary condition on the bottom is:

$$\vec{u}.\vec{n} = \vec{u}.(\hat{k} + \hat{i}h_x) = 0$$

or

$$w = -uh_x \quad \text{on} \quad z = -h(x), \tag{3.5}$$

while the surface condition is simply that:

$$w = 0 \quad \text{on} \quad z = 0, \tag{3.6}$$

if one assumes a rigid top.

The analysis of the problem posed by equations (3.1) to (3.3) with the conditions (3.5) and (3.6) is clearer if the equations are scaled. As before the idea is to balance the linear terms and have the nonlinear ones multiplied by a small parameter. For internal wave dynamics an appropriate time scale is given by the inverse of the Brunt-Vaisala frequency N. N is defined by:

$$N = \left(-\frac{g}{\rho_0}\,\frac{\partial \rho_s(z)}{\partial z}\right)^{1/2} \tag{3.7}$$

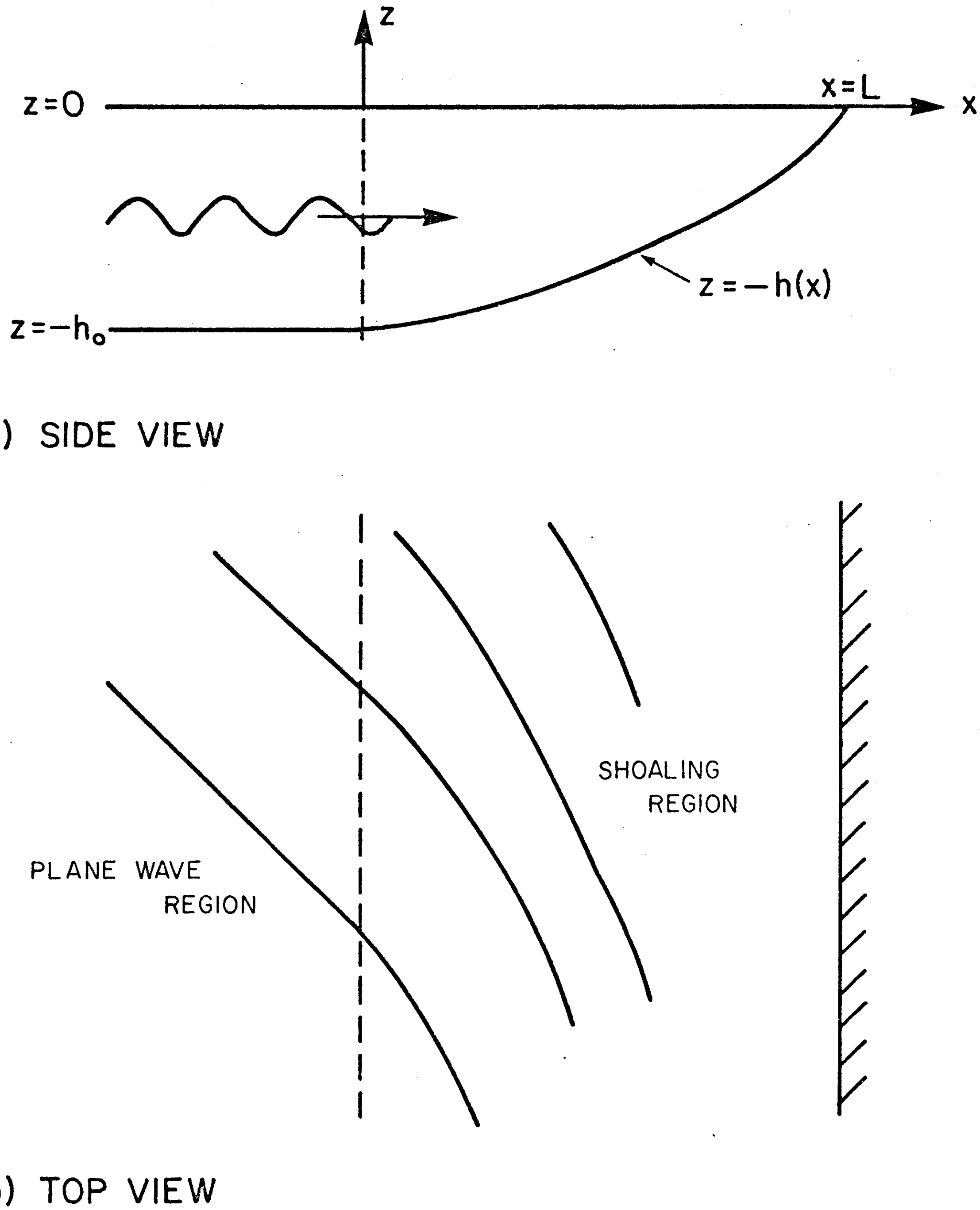



Figure 22. A pictorial representation of the shoaling internal wave problem.

where $\rho_0$ is a reference density of the fluid (the density at $z=0$ for instance) and $\rho_0 + \rho_s(z)$ is the undisturbed fluid density. The basic stratification will be assumed linear (i.e. $\rho_s(z)=-\Delta\rho\, z/H$) so that N is constant. If the wave amplitude in deep water, where $h(x)=h_0$, is $a_o$ then a velocity scale is $Na_o$. When the Boussinesq approximation is made in ignoring the variable stratification effects on the inertia, density and pressure perturbations arise whose scale can be determined from a balance with the linear terms in the equations.

There are two intrinsic length scales; one associated with the local wave dynamics which have a scale $|\vec{k}_o|^{-1}$ or the inverse of the deep water wavenumber magnitude, and the other related to bottom variations which have an influence on a length scale L, the total beach length. The technique used in the analysis will be essentially a WKB expansion in which the assumption is made that the bottom changes little over one wavelength. This restriction is expressed more clearly by the parameter

$$\alpha = |\vec{k}_o| L \ll 1 \tag{3.8}$$

which is a measure of the bottom slope. These considerations lead to the scaling procedure:

$$\begin{aligned} x &= |\vec{k}_o|^{-1} x', \\ \vec{u} &= Na_o \vec{u}', \\ p &= p_o + p_s(z) + \rho_o |\vec{k}_o|^{-1} N^2 a_o p'(x',y',z',t'), \\ \rho &= \rho_o + \rho_s(z) + \rho_o g^{-1} N^2 a_o \rho'(x',y',z',t'), \end{aligned} \tag{3.9}$$

and $$t = N^{-1}t'$$

where $$\nabla p_s(z) = -g(\rho_o + \rho_s(z))\hat{k}$$

and $\rho_o$ and $p_o$ are constant reference densities and pressures

Substitution of the above transformation into the equations of motion gives the set:

$$\frac{\partial \vec{u}'}{\partial t'} + \epsilon_o \vec{u}' \cdot \nabla' \vec{u}' + S\hat{k} \times \vec{u} = -\nabla' p' - \rho' \hat{k} \tag{3.10}$$

$$\frac{\partial \rho'}{\partial t'} + \epsilon_o \vec{u}' \cdot \nabla' \rho' - w' = 0 \tag{3.11}$$

$$\nabla' \cdot \vec{u}' = 0 \tag{3.12}$$

Two additional parameters have appeared. The strength of Coriolis relative to buoyancy forces relative to buoyancy forces is now measured by:

$$S = f/N \tag{3.13}$$

and, without the aspect ratio of the steady flow problem, this is generally a small quantity except in the deep ocean. The slope of the waves in deep water is measured by the Stokes parameter:

$$\epsilon_o = a_o |\vec{k}_o| \tag{3.14}$$

which is also found to multiply the nonlinear terms in the equations of motion.

In deriving equations (3.10) to (3.12) the Boussinesq approximation has lead to the dropping of terms like $\rho_s(z)u_t$

which have a magnitude of $\Delta\rho/\rho_0$ relative to the linear terms retained. $-\Delta\rho$ is the density difference between top and bottom. As the following analysis will be carried to $O(\epsilon_0^2)$, the results will be valid only if

$$1 > \epsilon_0^2 > \frac{\Delta\rho}{\rho_0}$$

a condition on both how small, and how large the wave slope can be. In fact, if a parameter measuring Boussinesq effects were included in the expansion scheme it would be $O(\Delta\rho/\rho_0)$ and the first order correction would still be a linear problem which could have no effect on the mean and would only influence the vertical structure of the waves. As far as the mean results are concerned, therefore, restriction (3.15) can be based on the density perturbation $\rho'(x',y',z',t')$ instead of $\rho_s(z)$ so that:

$$1 > \epsilon_0^2 > \frac{a_0}{h_0} \cdot \frac{\Delta\rho}{\rho_0} \tag{3.15}$$

The boundary conditions, from (3.5) and (3.6) are:

$$w' = 0 \text{ on } z' = 0, \tag{3.16}$$

and

$$w' = -\alpha u' h_X(X) \text{ on } z' = -h'(X). \tag{3.17}$$

A new stretched x-coordinate has been introduced which is defined by:

$$X = \alpha x \tag{3.18}$$

and indicates distances over which the bottom changes appreciably.

In the linear problem with a free surface the magnitude of the surface displacement relative to the interior particle movements can be shown to be of the order $\frac{\omega^2}{g|\vec{k}|}\left(\frac{N^2}{\omega^2}-1\right)^{1/2}$ (see Phillips p165) where $\omega$ is the wave frequency. For this motion to be a smaller effect than those of interest in this investigation then:

$$1 > \epsilon_0^2 > \frac{N^2}{g|\vec{k}|},$$

or

$$1 > \epsilon_0^3 > \frac{a_0}{h_0}\frac{\Delta\rho}{\rho_0}.$$

is a sufficient condition as $\omega^2\left(\frac{N^2}{\omega^2}-1\right)^{1/2} < N^2$. But, once again, the first correction accounting for this motion will be derived from a linear problem and cannot introduce mean effects. The above restriction can be reduced an order in $\epsilon_0$ when the nonlinear effects on the mean are considered and the condition reduces to (3.15).

After taking the time average of equations (3.10) to (3.12) one finds that: (dropping primes)

$$\epsilon_0 \nabla\cdot\overline{\vec{u}u} - S\overline{v} = -\overline{p}_x \tag{3.19}$$

$$\epsilon_0 \nabla\cdot\overline{\vec{u}v} + S\overline{u} = -\overline{p}_y \tag{3.20}$$

$$\epsilon_0 \nabla\cdot\overline{\vec{u}w} = -\overline{p}_z - \overline{\rho} \tag{3.21}$$

$$\epsilon_0 \nabla\cdot\overline{\vec{u}\rho} - \overline{w} = 0 \tag{3.22a}$$

$$\nabla\cdot\overline{\vec{u}} = 0 \tag{3.23a}$$

The primary goal of this chapter is to derive these mean quantities. For $\epsilon_0 \ll 1$ it appears that mean effects of $O(\epsilon_0)$ can be produced by the Reynolds stress terms $\nabla\cdot\overline{\vec{u}\vec{u}}$ and $\nabla\cdot\overline{\vec{u}\rho}$ forced by the O(1) wave motion. These stress terms contain x derivatives which give $O(\alpha)$ mean effects and z derivatives which are O(1). However, in the horizontal force balances, these O(1) stresses vanish and it is necessary to know the wave field to $O(\alpha)$ in order to calculate the Reynolds stresses. These $O(\alpha)$ quantities will be derived from a WKB expansion. For normal incidence the approach will be slightly more general giving additional information on the wavenumber variation and harmonic generation.

As Wunsch (1971) notes, the inclusion of rotation in the problem gives rise to special difficulties. The mean flow along the beach is arbitrary in this formulation and the Coriolis force produced by this current can support the Reynolds stress terms in equation (3.19) and reduce the necessary pressure gradient. In addition the Coriolis force acts in the identical manner as the bottom slope to inhibit mean Lagrangian drift as Hasselmann (1970) notes.

3.3. Normal Incidence - Plane Beach

For this highly specialized case all y-derivatives are neglected and a streamfunction $\phi(x,z,t)$ satisfying the continuity equation (3.12) can be defined according to:

$$\begin{aligned} u &= -\phi_z \\ w &= \phi_x . \end{aligned} \tag{3.22}$$

Using this streamfunction, the upslope and vertical momentum equations become:

$$\begin{aligned} -\phi_{zt} - Sv &= -p_x - \epsilon_o \nabla\cdot\vec{u}\, u \\ \phi_{xt} \quad &= -p_z - \rho - \epsilon_o \nabla\cdot\vec{u}\, w \end{aligned}$$

from which pressure can be eliminated to give:

$$\nabla^2\phi_t + Sv_z = -\rho_x - \epsilon_o J(\phi, \nabla^2\phi) \tag{3.23}$$

where $J(\phi, \nabla^2\phi)$ is the Jacobian determinant of $\phi$ and $\nabla^2\phi$ with respect to x and z. This equation gives the departures from a thermal wind balance due to time dependent and nonlinear processes. The alongslope momentum equation and the mass conservation relation in the streamfunction notation are:

$$\frac{\partial v}{\partial t} - S\phi_z = -\epsilon_o J(\phi, v) \tag{3.24}$$

and

$$\frac{\partial \rho}{\partial t} - \phi_x = -\epsilon_o J(\phi, \rho) \tag{3.25}$$

respectively. Equations (3.24) and (3.25) can be combined with equation (3.23) to obtain a single equation for $\phi(x,z,t)$ which contains a nonlinear dependence on $v(x,z,t)$ and $\rho(x,z,t)$. This is:

$$\nabla^2\phi_{tt} + S^2\phi_{zz} + \phi_{xx} = \epsilon_o\left\{\frac{\partial}{\partial x}J(\phi,\rho) - \frac{\partial}{\partial t}J(\phi,\nabla^2\phi) + S\frac{\partial}{\partial z}J(\phi,v)\right\} \tag{3.26}$$

Once $\phi(x,z,t)$ is determined at any order in $\epsilon_o$, $\rho(x,z,t)$ and $v(x,z,t)$ can be determined from (3.24) and (3.25). The boundary conditions (3.16) and (3.17) become simply:

$$\phi = 0 \text{ at } z = \begin{pmatrix} 0 \\ -h(X) \end{pmatrix} \tag{3.27}$$

Under the assumption that $\alpha$ and $\epsilon_o$ are small an approximate solution can be obtained using the following generalized WKB expansion of $\phi$, $\rho$, and v:

$$\phi(x,z,t) = \sum_{n=1}^{\infty}\lambda^{n-1}\sum_{m=-n}^{n}P^{(n,m)}(X,z)e^{im(\psi-\omega t)} \tag{3.28}$$

$$\rho(x,z,t) = \sum_{n=1}^{\infty}\lambda^{n-1}\sum_{m=-n}^{n}R^{(n,m)}(X,z)e^{im(\psi-\omega t)} \tag{3.29}$$

and

$$v(x,z,t) = \sum_{n=1}^{\infty}\lambda^{n-1}\sum_{m=-n}^{n}V^{(n,m)}(X,z)e^{im(\psi-\omega t)} \tag{3.30}$$

$\lambda$ is an ordering parameter which is used so that $\epsilon_o$ and $\alpha$ can vary independently. It can be considered to be representative of the order of magnitude of $\epsilon_o$ and $\alpha$ . In order to evaluate the combined effect of both bottom and wave slopes they will be considered to be of the same order.

The phase function $\psi(x)$ is related to the horizontal wavenumber k(X) by:

$$k(X) = \frac{\partial\psi}{\partial x} \tag{3.31}$$

or
$$\psi(x) = \frac{1}{\alpha}\int_0^{\alpha x} k(X)\,dX$$

$k(X)$ and, therefore, $\psi(x)$ can be taken to be real because any complex part is included in the amplitude function. In addition the wavenumber has an expansion which depends on wave and bottom slope of the form:

$$k(X) = \sum_{n=0}^{\infty} \lambda^{2n} k^{(2n)}(X) \tag{3.32}$$

For convenience, only the even powers of n are used. With $P^{(n,m)}(X,z)$ treated as a complex function the imaginary part is equivalent to a wavenumber and some redundance exists in the above expansion procedure. To remove this it will be necessary to make an arbitrary choice in the amplitude function for odd powers of n in sequence of problems that is discussed in the following analysis.

By making the definitions:

$$\begin{aligned} V^{(n,-m)}(X,z) &= V^{*(n,m)}(X,z), \\ P^{(n,-m)}(X,z) &= P^{*(n,m)}(X,z), \\ \text{and} \quad R^{(n,-m)}(X,z) &= R^{*(n,m)}(X,z). \end{aligned} \tag{3.33}$$

where the * denotes complex conjugation, the quantities represented by the expansions (3.28), (3.29) and (3.30) will be real.

Chu and Mei (1970) have used a similar scheme to that outlined above in order to study slowly varying surface waves.

Their problem generated a series of boundary value problems in two unknowns, the surface displacement and streamfunction. On substitution of the above expansion into (3.24), (3.25), and (3.26) a sequence of boundary value problems in three unknowns is generated. The problems at each order in $\lambda$ up to $\lambda^2$ and each harmonic number m up to m=2 are summarized below.

<u>$O(\lambda^0)$; m=0:</u> One of the underlying assumptions of this analysis is that mean effects, represented by the zeroth harmonic, are generated by the small nonlinearities in the equations interacting with the small bottom slope, and therefore these quantities must be of higher order than the fundamental wave field. Because of this:

$$P^{(1,0)} = R^{(1,0)} = V^{(1,0)} = 0 \; . \tag{3.34}$$

<u>$O(\lambda^0)$; m=1:</u> The solutions at this level are the usual internal wave modes in a uniform waveguide. Equation (3.26) becomes:

$$-\omega^2\left(-k^{(0)2}P^{(1,1)} + \frac{\partial^2 P^{(1,1)}}{\partial z^2}\right) + S^2\frac{\partial^2 P^{(1,1)}}{\partial z^2} - k^{(0)2}P^{(1,1)} = 0$$

or

$$P_{zz}^{(1,1)} + \frac{k^{(0)2}}{C^2}P^{(1,1)} = 0 \tag{3.35}$$

where

$$C^2 = \frac{\omega^2 - S^2}{1-\omega^2} \tag{3.36}$$

after Wunsch (1969).

The boundary conditions from equation (3.20) are:

$$P^{(1,1)} = 0 \quad \text{at} \quad z = \begin{pmatrix} 0 \\ -h(X) \end{pmatrix} . \tag{3.37}$$

and the solution of (3.35) satisfying these conditions is:

$$P^{(1,1)}(X,z) = A^{(1,1)}(X) \sin \frac{k^{(0)}z}{c} \tag{3.38}$$

with
$$k^{(0)}(X) = \frac{c\ell\pi}{h(X)} \quad (\ell = +\text{ve integer}). \tag{3.39}$$

$A^{(1,1)}(X)$ is the wave amplitude and its dependence on the large scale is determined at $O(\lambda)$ where energy conservation effects specify how the amplitude must change with variations in the bottom depth. $R^{(1,1)}$ and $V^{(1,1)}$ are found in terms of $P^{(1,1)}$ from (3.24) and (3.25) to be:

$$R^{(1,1)}(X,z) = -\frac{k^{(0)}}{\omega} P^{(1,1)}(X,z), \tag{3.40}$$

and
$$V^{(1,1)}(X,z) = i\frac{S}{\omega} P_z^{(1,1)}(X,z). \tag{3.41}$$

The only influence of the sloping bottom that has been determined at this order is the effect on the wavenumber. From (3.39) $k^{(0)}$ is seen to have a dependence that is inversely proportional to depth and waves shorten as they move into shallower water.

<u>$O(\lambda^0)$; m=-1:</u> From equation (3.34) these solutions are simply:

$$P^{(1,-1)}(X,z) = P^{*(1,1)}(X,z) = P^{(1,1)}(X,z),$$
$$R^{(1,-1)}(X,z) = R^{*(1,1)}(X,z) = R^{(1,1)}(X,z), \tag{3.42}$$

and
$$V^{(1,-1)}(X,z) = V^{*(1,1)}(X,z) = -V^{(1,1)}(X,z).$$

O($\lambda$); m=0: In the O($\lambda$) series of problems the wave slope and bottom slope influences are still separate. They cannot interact until the next order in $\lambda$. For the $0^{th}$ harmonic equation (3.24) yields:

$$-S\lambda P_z^{(2,0)} = -\epsilon_0 \{ ik^{(0)} P^{(1,1)} V_z^{(1,-1)} - ik^{(0)} P_z^{(1,1)} V^{(1,-1)} + ik^{(0)} P^{(1,-1)} V_z^{(1,1)} + ik^{(0)} P_z^{(1,-1)} V^{(1,1)} \}$$

$$= -S \frac{k^{(0)}}{\omega} \epsilon_0 \frac{\partial^2 P^{2(1,1)}}{\partial z^2} \qquad (3.43)$$

from (3.42). Integrating this equation and having the result satisfy the top and bottom boundary conditions, the O($\lambda$) mean streamfunction is found to be:

$$P^{(2,0)}(X,z) = \frac{\epsilon_0}{\lambda} \frac{k^{(0)}}{\omega} \frac{\partial}{\partial z} P^{2(1,1)}$$

$$= \frac{\epsilon_0}{\lambda} \frac{k^{2(0)}}{c\,\omega} A^{2(1,1)}(X) \sin \frac{2k^{(0)}z}{c} \qquad (3.44)$$

If the z-derivative of this expression is taken one derives the following expression for the mean Eulerian upslope current:

$$\bar{u}_E(X,z) = -2\epsilon_0 \frac{k^{3(0)}}{c^2 \omega} A^{2(1,1)} \cos \frac{2k^{(0)}z}{c} + \cdots \qquad (3.45)$$

As Wunsch (1971) points out, there can be no net motion of a particle along an isopycnal as these constant density surfaces intersect the beach where no mixing has been allowed for in this model. Net motion is measured by the averaged Lagrangian velocity $\bar{\vec{u}}_L(X,z)$. Wunsch shows that, when slope effects are considered, an equal but opposite "Stokes velocity" $\bar{\vec{u}}_S(X,z)$ is present in the waves such that:

$$\overline{u}_L(X,z) = \overline{u}_E(X,z) + \overline{u}_S(X,z)$$
$$= 0 \ . \qquad (3.46)$$

The mean Stokes velocity is given in scaled form by:

$$\overline{\vec{u}}_s(X,z) = \epsilon_o \overline{\int_0^t \vec{u}(x,t')\,dt' \cdot \nabla \vec{u}(x,t)} \qquad (3.47)$$

where $\vec{u}(x,t)$ is the Eulerian velocity field. It can be easily shown for the O(1) streamfunction solution of (3.38) that to $O(\epsilon_o)$, $\overline{u}_S(X,t) = -\overline{u}_E(X,t)$.

At this level of approximation the mean Eulerian current is not generated in response to the bottom slope influence but by rotational effects. The $O(\lambda)$ mean streamfunction $P^{(2,0)}(X,z)$ is found in equation (3.43) by a balance between the Coriolis force produced by $\overline{u}_E(X,z)$ and the along shore Reynolds stress. A mean Lagrangian velocity is forced to vanish by this balance in a manner that has been discussed by Hasselmann (1970). Note that S and therefore the Coriolis parameter vanishes in the final expression and the limit of no rotation is, apparently, a singular one. It will be shown at $O(\lambda^2)$ that a similar problem develops for $P^{(2,0)}(X,z)$ from the density conservation equation (3.25). At this order the sloping bottom constrains the motion to have no Lagrangian velocity and the identical form of equation (3.44) for $P^{(2,0)}(X,z)$ is found.

Equation (3.23) at $O(\lambda)$ for the mean yields:

$$V_z^{(2,0)} = 0$$

or

$$V^{(2,0)} = V^{(2,0)}(X).$$

This $O(\lambda)$ barotropic flow must be balanced in the upslope momentum equation by an $O(1)$ mean pressure which varies on the X length scale to produce an $O(\alpha)$ pressure gradient. As the assumption has been made that no $O(1)$ quantities can be forced by the fundamental wave motion, the choice is made that:

$$V^{(2,0)}(X) = 0 \,, \qquad (3.48)$$

in order that the $O(1)$ mean pressure vanish. By keeping $V^{(2,0)}$ nonzero it is possible to study the interaction of the waves with a small geostrophic current flowing perpendicular to them. This complication will not be pursued.

<u>$O(\lambda)$; m=1:</u> As the mean (m=0) contributions at $O(\lambda^0)$ were assumed zero and the nonlinear term at this level is a product of $O(\lambda^0)$, m=0 and $O(\lambda^0)$, m=1 terms there is no nonlinear effect in this problem. Only slopingbottom influences are felt. Equation (3.26) becomes:

$$-\omega^2\left(-k^{2(0)}p^{(2,1)} + \frac{\partial^2 p^{(2,1)}}{\partial z^2}\right) + S^2\frac{\partial^2 p^{(2,1)}}{\partial z^2} - k^{2(0)}p^{(2,1)}$$
$$= \frac{\alpha}{\lambda}\left\{-2ik^{(0)}P_X^{(1,1)} - ik_X^{(0)}P^{(1,1)}\right\}$$

which, on substitution for $P^{(1,1)}$, can be rewritten:

$$P_{zz}^{(2,1)} + \frac{k^{2(0)}}{c^2}P^{(2,1)} = \frac{i\alpha}{c^2\lambda}\left\{\left(2k^{(0)}A_X^{(1,1)} + k_X^{(0)}A^{(1,1)}\right)\sin\frac{k^{(0)}z}{c} \right. \qquad (3.49)$$
$$\left. -2k^{(0)}A^{(1,1)}\frac{k^{(0)}}{c}\frac{h_X}{h}\cos\frac{k^{(0)}z}{c}\right\}\left(\frac{\omega^2}{\omega^2 - S^2}\right)$$

From (3.27) the boundary conditions are:

$$P^{(2,1)} = 0 \quad \text{at} \quad z = \begin{pmatrix} 0 \\ -h(X) \end{pmatrix}.$$

The general solution to this problem can be written in the form:

$$P^{(2,1)}(X,z) = A^{(2,1)}(X) \sin \frac{k^{(0)}z}{c} + \text{particular sol.}$$

However, $A^{(2,1)}(X)$ can be chosen to vanish for the same reason that $k(X)$ was expanded in even powers of $\lambda$. This part of the general solution is contained in the wavenumber expansion. The particular solution is found to be:

$$P^{(2,1)}(X,z) = \frac{1}{2}\cdot\frac{i}{c^2}\cdot A^{(1,1)}\cdot\frac{\alpha}{\lambda}\cdot k_X^{(0)}\cdot z^2 \cdot \sin\frac{k^{(0)}z}{c}\cdot\left(\frac{\omega^2}{\omega^2 - S^2}\right) \tag{3.50}$$

In the process of satisfying the boundary conditions with the general solution to (3.49) one finds that a constraint on $A^{(1,1)}(X)$ is imposed. This is:

$$A_X^{(1,1)}(X) = 0,$$

or

$$A^{(1,1)}(X) = \text{constant} = A^{(1,1)}. \tag{3.51}$$

The amplitude of the lowest order streamfunction does not change as the wave progresses onshore; but the horizontal velocity whose amplitude is $k^{(0)}A^{(1,1)}/c$ from (3.22) will increase as the inverse of the depth. This behavior could be anticipated from energy conservation arguments. Note also that $P^{(2,1)}(X,z)$ has a $z^2$ dependence on depth and therefore the motion intensifies toward the bottom, a property that Wunsch (1969) found in his wedge solutions.

$R^{(2,1)}(X,z)$ and $V^{(2,1)}(X,z)$ can be found from equations

(3.24) and (3.25) in terms of $P^{(2,1)}(X,z)$ and $P^{(1,1)}(X,z)$ to be:

$$R^{(2,1)}(X,z) = -\frac{k^{(0)}}{\omega} P^{(2,1)} + \frac{i\alpha}{\omega\lambda} P_X^{(1,1)} \tag{3.52}$$

and

$$V^{(2,1)}(X,z) = \frac{iS}{\omega} P_z^{(2,1)}. \tag{3.53}$$

<u>O( $\lambda$ ); m=2:</u> At this order the fundamental wave motion interacts with itself to produce a second harmonic. Substitution of the O( $\lambda^{\circ}$ ) solutions into (3.26) for m=2 gives:

$$-4\omega^2\left(-4k^{2(0)}P^{(2,2)} + \frac{\partial^2 P^{(2,2)}}{\partial z^2}\right) + S^2\frac{\partial^2 P^{(2,2)}}{\partial z^2} - 4k^{2(0)}P^{(2,2)} = 0$$

or

$$P_{zz}^{(2,2)} + \frac{4k^{2(0)}(1-4\omega^2)}{4\omega^2 - S^2} P^{(2,2)} = 0 \tag{3.54}$$

and the nonlinea correlations vanish identically. With the top and bottom conditions to be satisfied, the only eligible solution to (3.54) is the trivial one:

$$P^{(2,2)}(X,z) = 0. \tag{3.55}$$

In equation (3.25) the nonlinear terms also vanish so that:

$$R^{(2,2)}(X,z) = -\frac{k^{(0)}}{\omega}P^{(2,2)}(X,z) = 0 \tag{3.56}$$

while there is a nonlinear correlation in equation (3.24) and:

$$v^{(2,2)}(X,z) = \frac{iS}{2\omega} P_z^{(2,2)} - \frac{\epsilon_0}{\lambda}\frac{iS}{2\omega^2}\left(\frac{k^{(0)}}{c}\right)^2 k^{(0)} A^{2(1,1)}$$
$$= -\frac{i}{2}\frac{\epsilon_0}{\lambda} S k^{(0)}\left(\frac{k^{(0)}A^{(1,1)}}{c\omega}\right)^2. \qquad (3.57)$$

Although there is no second harmonic for the density and streamfunction at this level of approximation, one exists for the longshore velocity component and the particle displacement function as will be shown in the discussion chapter.

<u>$O(\lambda^2)$; m=0:</u> Instead of dealing with equation(3.26) for this mean calculation, it is more convenient to use equations (3.23), (3.24), and (3.25) which were used to construct (3.26). Equation (3.25) is, once again, an equation determining $P^{(2,0)}$ which was determined at a lower order by (3.44). However, at this order, it is the sloping bottom influence and not the Coriolis force which constrains the motion to having no net Lagrangian drift. In the averaged form of equation (3.25) at $O(\lambda^2)$ the slope parameter $\alpha$ appears on both sides and then cancels in the same manner as the rotational parameter S did at $O(\lambda)$, m=0.

The Jacobian $J(\phi, v)$ vanishes at this order for m=0 in equation (3.24) because $P^{(2,0)}=0$, and $\phi$ and v are in quadrature at $O(\lambda)$, m=±1. In order to satisfy the top and bottom boundary constraints one finds that:

$$P^{(3,0)}(X,z) = 0 \qquad (3.58)$$

Equation (3.23) gives the departures from a thermal wind relationship. Expanding the nonlinear term and keeping terms of O( $\lambda^2$) for the zeroth harmonic one finds that:(after some reduction)

$$SV_z^{(3,0)} = -\frac{\alpha}{\lambda} R_X^{(2,0)} + \frac{\epsilon_0 \alpha}{\lambda^2} \left(1 + \frac{1}{c^2}\right) \frac{\partial}{\partial X} \left( k^{2(0)} P^{(1,1)} P_z^{(1,1)} \right) \qquad (3.59)$$

This equation through the term $R_X^{(2,0)}$ determines the slope of the isopycnals. However, there is a degeneracy which cannot be removed with this treatment of the problem. The balance in (3.59) is between Coriolis forces, pressure gradient and the Reynolds stress produced by the fundamental wave field. Unfortunately, another relationship between $V^{(3,0)}$ and $R^{(2,0)}$ is not available, and either of these terms can support in varying degrees the onshore Reynolds stress. If $SV_z^{(3,0)}$ is small, as it seems to be for the Bermuda observations, then (3.59) can be integrated to give:

$$R^{(2,0)}(X,z) = \frac{\epsilon_0}{2} \left(1 + \frac{1}{c^2}\right) k^{2(0)} \frac{\partial}{\partial z} P^{2\,(1,1)} \qquad (3.60)$$

A function of z resulting from the integration has been set equal to zero and, as a result, (3.60) gives a mean density shift over the flat bottom region. This is another indeterminancy which cannot be removed without studying the initial value problem. A related indeterminant function arises in the calculation of the mean isopycnal displacement. Thorpe (1968) defines this mean quantity to be zero to solve the problem. In this study with a sloping bottom the function of z can only be chosen at one depth and results in nonzero values of whichever quantity was chosen to vanish at depths different from this. As $R^{(2,0)}(X,z)$ grows as the inverse of the square of the depth in the shoaling region any assumption about its size in the deeper fluid soon loses relevance.

$O(\lambda^2)$; m=1: At this order the first wavenumber correction enters. The present capability for observing internal waves in the ocean is such that the $O(\lambda^0)$ dependence of $k^{(0)}(X)$ on depth cannot be confirmed. In addition, there are many modes and frequencies present in the ocean leading to interactions affecting the $O(\lambda^2)$ results. The most reasonable application of the wavenumber correction would be in laboratory experiments where S is generally very small and experiments conducted over such short periods of time that any influence of the earth's rotation will be negligible.

When S=0 equation (3.26) gives: (after considerable reduction)

$$\begin{aligned} \frac{\partial^2 P^{(3,1)}}{\partial z^2} + \frac{k^{2(0)}}{c^2} P^{(3,1)} = \frac{1}{c^2} \Big\{ & \Big( -2k^{(0)}k^{(2)}A^{(1,1)} + 4\left(\frac{\epsilon_0}{\lambda}\right)^2 \frac{k^{6(0)}}{\omega^2 c^2} A^{3(1,1)} \Big) \cdot \\ & \cdot \sin\frac{k^{(0)}z}{c} - 4\left(\frac{\epsilon_0}{\lambda}\right)^2 \frac{k^{6(0)}}{\omega^2 c^2} \sin\frac{3k^{(0)}z}{c} - \\ & - \frac{A^{(1,1)}}{c^2}\left(\frac{\alpha}{\lambda}\right)^2 \left(\frac{1}{2}(k_X^{(0)})^2 + k^{(0)}k_{XX}^{(0)}\right) z^2 \sin\frac{k^{(0)}z}{c} - \left(\frac{\alpha}{\lambda}\right)^2 A^{(1,1)} k^{(0)} \left(k_X^{(0)}\right)^2 z^3 \cos\frac{k^{(0)}z}{c} \Big\} \end{aligned} \tag{3.61}$$

The solution of this equation is:

$$\begin{aligned} P^{(3,1)}(X,z) = {} & A^{(3,1)}(X) \sin\frac{k^{(0)}z}{c} + (b_1 z + b_2 z^3) \cos\frac{k^{(0)}z}{c} \\ & + (b_3 z^2 + b_4 z^4) \sin\frac{k^{(0)}z}{c} + b_5 \sin\frac{3k^{(0)}z}{c} \end{aligned} \tag{3.62}$$

where:

$$b_1 = \frac{k^{(2)}A^{(1,1)}}{c} - 2\left(\frac{\epsilon}{\lambda}\right)^2 \frac{A^{3(1,1)}k^{5(0)}}{c^3\omega^2},$$

$$b_2 = \left(\frac{\alpha}{\lambda}\right)^2 \frac{A^{(1,1)}}{c^3} \frac{k_{XX}^{(0)}}{6},$$

$$b_3 = 0 \,,$$
$$b_4 = -\frac{1}{8}\left(\frac{\alpha}{\lambda}\right)^2 \frac{A^{(1,1)}\left(k_X^{(0)}\right)^2}{c^4} \,,$$
$$b_5 = \frac{4}{9}\left(\frac{\epsilon}{\lambda}\right)^2 \frac{A^{3(1,1)}\,k^{4(0)}}{c^2\,\omega^2} \,. \qquad (3.63)$$

The function $A^{(3,1)}(X)$ can be determined at the next order.

Solution (3.62) must satisfy the vanishing constraint at top and bottom implying that:

$$b_1 + h^2 b_2 = 0$$

and therefore:

$$k^{(2)} = 2\left(\frac{\epsilon_0}{\lambda}\right)^2 \frac{A^{2(1,1)}\,k^{5(0)}}{c^3\,\omega^2} - \left(\frac{\alpha}{\lambda}\right)^2 \frac{h^2\,k_{XX}^{(0)}}{6c^2} \qquad (3.64)$$

For a linearly sloping beach of the form $h(x)=h_o-\alpha x$:

$$k_{XX}^{(0)} = 2\,k^{(0)}\left(\frac{h_X}{h}\right)^2$$

and:
$$k(X) = k^{(0)}\left(1 + 2\epsilon_0^{\,2}\frac{A^{2(1,1)}k^{4(0)}}{c^3\,\omega^2} - \alpha^2\frac{(h_X)^2}{3c^2} + O(\lambda^3)\right) \qquad (3.65)$$

The effect of the nonlinearities is to further shorten the waves while bottom slope tends to lengthen the waves at $O(\alpha^2)$. Wave slope influences are potentially much more significant than bottom variations as their strength increases as the inverse of the fourth power of depth relative to the constant beach slope correction. Unfortunately, for these purposes, Cacchione (1970) measured the length of his waves over the slope only to a point where the local wave slope was about .15 and the nonlinear correction developed above would contribute an error of only 2% which is insignificant.

The higher harmonics at this order, as well as higher orders in $\lambda$ , can be calculated in a similar fashion to the above. However, not only does the algebra become more complicated but the results, through the neglected viscous effects and other assumptions, become insignificant.

## 3.4. Discussion

The quantity of greatest interest, as far as the data of chapter (1) is concerned, is the mean isopycnal depth change over the beach forced through the Reynolds stress mechanism. Along an isopycnal density is constant by definition and, if $\eta(x',t';z_o')$ is the scaled departure from the undisturbed isopycnal depth $z_o'$, then:

$$z' = z_o' + \epsilon_o \eta'(x',t';z_o') \tag{3.66}$$

is the actual depth and,

$$\begin{aligned}\rho &= \text{constant} \\ &= \rho_o + \rho_s\left(\frac{z'}{k_o}\right) + \frac{a_o N^2 \rho_o}{g}\rho'(x',z',t')\end{aligned} \tag{3.67}$$

from equation (3.9).

Substituting (3.66) into (3.67) and then expanding $\rho_s\left(\frac{z'}{k_o}\right)$ and $\rho'(x',z',t')$ in a Taylor series expansion about $z_o'$ one finds that:

$$\begin{aligned}\text{constant} = \rho_o + \rho_s\left(\frac{z_o'}{k_o'}\right) + \left.\frac{\partial\rho_s}{\partial z}\right|_{z_o'} a_o\eta' + \left.\frac{\partial^2\rho_s}{\partial z^2}\right|_{z_o'} \frac{(a_o\eta')^2}{2} + \cdots \\ + \frac{a_o N^2\rho_o}{g}\left(\rho'(x',z_o',t') + \left.\frac{\partial\rho'}{\partial z'}\right|_{z_o'}\epsilon_o\eta + \cdots\right)\end{aligned}$$

Using the knowledge that $\left.\frac{\partial\rho_s}{\partial z}\right|_{z_o} = -\frac{\rho_o N^2}{g}$ and $\left.\frac{\partial^2\rho_s}{\partial z^2}\right|_{z_o} = 0$ for the linear stratification then:

$$\eta'(x',t';z_o') = \rho'(x',z_o',t') + \epsilon_o \left.\frac{\partial\rho'}{\partial z'}\right|_{z_o'} \eta'(x',t';z_o') + O(\epsilon_o^2) \tag{3.68}$$

Now $\eta'(x',t';z_o')$ can be expanded in the WKB power series in $\lambda$ according to:

$$\eta'(x',t';z_o') = \sum_{n=1}^{\infty} \lambda^{n-1} \sum_{m=-n}^{n} N^{(n,m)}(X';z_o')e^{im(\psi-\omega't')} \tag{3.69}$$

from which one deduces that:

$$N^{(1,1)}(X';z_o') = R^{(1,1)}(X',z_o') = -\frac{k'^{(o)}}{\omega'} P^{(1,1)}(X';z_o') \tag{3.70}$$

$$N^{(2,0)}(X';z_o') = R^{(2,0)}(X',z_o') + \frac{\epsilon_o}{\lambda}\left(R_z^{(1,1)}N^{(1,-1)} + R_z^{(1,-1)}N^{(1,1)}\right) \tag{3.71}$$

and

$$N^{(2,2)}(X';z_o') = \frac{\epsilon_o}{\lambda} R_z^{(1,1)} N^{(1,1)} \quad . \tag{3.72}$$

for the lowest order contributions to the mean, first harmonic, and second harmonic respectively.

It is interesting that there is an O($\lambda$) contribution to the second harmonic in the isopycnal displacement function and that none exists for the density perturbation and stream-function. In a laboratory experiment, therefore, it would depend crucially on the quantity measured whether or not a strong second harmonic would be observed. Thorpe (1968) arrives at a similar result.

At the Bermuda shore the velocity shear is of the order of 50 cm/sec in 1km of fluid. Scaling this with respect to a 10m wave of length 1km progressing in a stratified medium with $N=2\times10^{-3}$ sec$^{-1}$ then the scaled shear $v'_{z'}=.4$. As the parameter S is small the quantity $Sv'_{z'}$ will be small relative to the other terms in equation (3.59) provided $Sv'_{z'} < \epsilon_o \alpha(1+1/c^2)$. A better indication of the nonlinear effect in equation (3.59) is the local Stokes parameter $\epsilon$ which is defined by:

$$\epsilon = \epsilon_0 \left( \frac{h_0}{h(X)} \right)^2 = a\,k^{(0)} \tag{3.73}$$

At some point on the beach, no matter what the deep water value of $\epsilon$ is, it will increase to the point that $\epsilon\alpha(1+1/c^2) > Sv_{z'}'$, and the Reynolds stress will be mostly supported by a density gradient.

Ignoring the Coriolis term in (3.59) and using (3.60), equation (3.72) becomes:

$$N^{(2,0)}(X';z_0') = -\frac{\epsilon_0}{\lambda}\frac{k'^{3(0)}}{c\,\omega'^2} A^{2(1,1)} \sin\frac{2k'^{(0)}z_0'}{c} \tag{3.74}$$

with

$$c^2 = \frac{\omega^2}{1-\omega^2} \tag{3.75}$$

If the actual lowest order wave amplitude is $a'(X')$, then:

$$a'(X') = -\frac{2k'^{(0)}(X')}{\omega'} A^{(1,1)} \tag{3.76}$$

showing that the wave amplitude increases in inverse proportion to depth through its dependence on $k^{(0)}(X')$. Substituting into (3.74) from (3.76) and then putting the result in dimensional form, the mean isotherm position is:

$$\overline{\eta}(X;z_0) = -\frac{1}{4}n\pi\frac{a_0^2}{h_0}\sin\frac{2n\pi z}{h(X)} \cdot \left(\frac{h_0}{h(X)}\right)^3 \tag{3.77}$$

In using this expression to calculate $\overline{\eta}(X)$, care must be taken not to extend the result too far up the beach where $\overline{\eta}(X;z_0)$ can increase without limit. The true measure of the magnitude of the nonlinear effects is the local Stokes parameter of equation (3.73) and a better stipulation is that this be small for the results to be valid.

Suppose, for example, that a steady train of internal waves are shoaling from a region of uniform depth 3km (i.e. $h_o$=3km) onto a linear beach of slope 0.2 (i.e. h(x)=3000-x/5 m) which is typical of Bermuda. If the initial wavelength is 3km and the amplitude is 10m then $\epsilon_o$=.02. Taking the largest value of the local Stokes parameter for which these results can be used to be $\epsilon$ =.5, then the ratio of $h_o$ to h(x) at this point is:

$$\left.\frac{h_o}{h(X)}\right|_{\epsilon=.5} = \left(\frac{.5}{.02}\right)^{1/2} = 5$$

and the local wave amplitude is:

$$a\Big|_{\epsilon=.5} = a_o\left(\frac{h_o}{h(X)}\right)^2 = 250\ m.$$

At this point equation (3.77) yields:

$$\bar{\eta}(X;z_o)\Big|_{\epsilon=.5} = -3.3\sin\left(\frac{2\pi z}{h(X)}\right) m.$$

for the first mode.

The amplitude of the mean displacement seems somewhat insignificant in comparison with the local wave amplitude but $\bar{\eta}$(X) increases as $h^{-3}$(X) while the wave amplitude grows as $h^{-2}$(X). As the wave progresses onshore the mean effects become increasingly significant compared to the amplitude. If equation (3.77) is pushed as far as $\epsilon$ =1 where it is clearly no longer valid as higher orders are equal in magnitude, isopycnal shifts of over 10m are predicted.

Another restriction on the application of the results concerned the use of the Boussinesq approximation and rigid surface. For the mean quantities to be valid this restriction is given by equation (3.15). For this example $\epsilon_0 \sim 10^{-5}$ while $\frac{a_0}{h_0}\left|\frac{\Delta\rho}{\rho_0}\right| \sim 3\text{x}10^{-6}$ and the stipulation is satisfied. Further upslope a better measure of the nonlinear effects is the local wave slope parameter which is larger than $\epsilon_0$ and the condition is even better satisfied.

A plot of equation (3.77) for the example under discussion is given in figure (23). The important difference between the nature of these isopycnal shifts and those investigated in chapter (2) is that, at any position along the beach, the displacements are in both directions and of equal magnitude. That is, in a vertical column of fluid over the beach the isopycnals are displaced in the mean both up and down to the same extent. In addition, for a high mode wave, the mean shifts will fluctuate rapidly in sign as the corner is approached. Bermuda observations of cruise Gosnold 144 indicate displacements which are vertically all downward as the island is approached in the region of apparent mixing (see figures 8-14) making it unlikely that this is the mechanism. The steady flow arguments are more reasonable.

However, during the first two weeks of this cruise (stations 1-20) the flow was somewhat weaker and from another direction. Although the current was to the southwest instead of the east a great deal of microstructure was still evident off

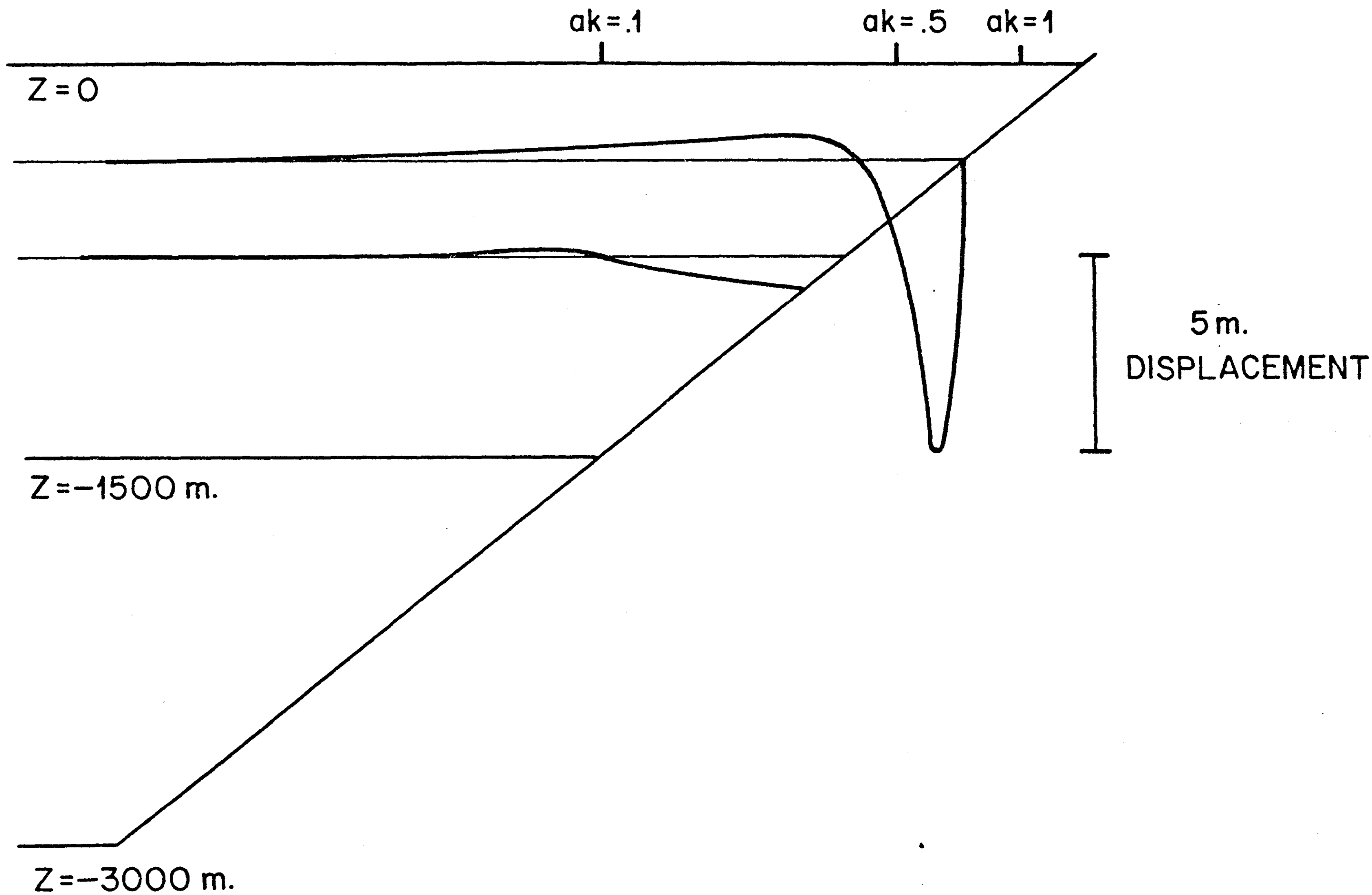
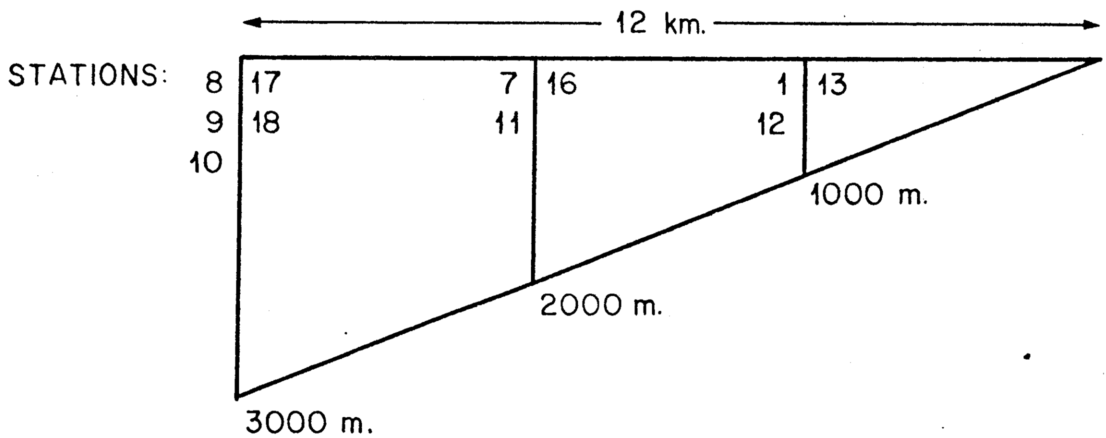



Figure 23. The mean isopycnal shift for a first mode imput wave of $a_0$= 10m. incident from a region with $h_0$=3km.

the northern coast. Although it is not possible to conclude a great deal about the average conditions from such a limited sample of stations an attempt was made to look for an isopycnal shift in this data. To reduce the contamination of the data due to the presence of the waves themselves the temperature and salinity values for the stations on about the same depth contour in the mixed area were averaged and the depth change of selected temperature and salinity values calculated with respect to the deepest contour. Results of this calculation are given in figure (24). Displacements in both directions that increase toward shore are noted. The results are probably not statistically significant because of the low number of data points but they are encouraging.

The picture of what is happening, according to this theory, is the following. Internal gravity waves, produced by some source to the north of the island are incident upon the northern slope. As they progress onshore, their steepness increases through the growth of the amplitude and wavenumber. Reynoldsstresses, produced by the nonlinear terms interacting with the bottom to inhibit mean Lagrangian flow along the isopycnals, cause a set up and set down of these constant density surfaces. Finally the waves become so steep that they are unstable and they break, mixing the fluid. This mixed fluid is then diffused back out from the shallow region to form step like features on the temperature and salinity traces. This mixing can also break down the constraint that there be no mean Lagrangian flow along isopycnals that end in the breaking




| DEPTH | AVERAGED | | AVERAGE SHIFT (m.) | |
|---|---|---|---|---|
| | TEMP. | SALINITY | 2000 m. | 1000 m. |
| 125 | 18.60 | 35.57 | −2±2 | −18±8 |
| 250 | 17.70 | 35.50 | 8±4 | −19±5 |
| 375 | 17.19 | 36.43 | −13±7 | 5±10 |
| 500 | 16.92 | 36.39 | 10±13 | 54±12 |
| 750 | 10.99 | 35.38 | 4±7 | 20±4 |

Figure 24. Data from the mixed region on GOSNOLD-144 showing a mean shift in the isopycnals.

area. In this way mixed fluid could be advected as well as diffused away from the mixing zone. A process such as this has been observed by Cacchione (1970) in his laboratory study of this problem.

As similar mixing regions were not observed in other areas around the island one has to postulate an anisotropic wave field, in particular a generating source to the north. Perhaps internal waves are produced by some process involving the degradation of the Gulf Stream.

## 3.5. Oblique Incidence - Longshore Currents

The normal incidence constraint on the analysis of section (3.3) can be removed if one is satisfied with less complete results. Ignoring rotation once again, equations (3.11) to (3.13) become:

$$\frac{\partial u}{\partial t} + \epsilon_o \nabla\cdot\vec{u}\, u = -p_x \tag{3.86}$$

$$\frac{\partial v}{\partial t} + \epsilon_o \nabla\cdot\vec{u}\, v = -p_y \tag{3.87}$$

$$\frac{\partial w}{\partial t} + \epsilon_o \nabla\cdot\vec{u}\, w = -p_z - \rho \tag{3.88}$$

$$\frac{\partial \rho}{\partial t} + \epsilon_o \nabla\cdot\vec{u}\, \rho - w = 0 \tag{3.89}$$

$$\nabla\cdot\vec{u} = 0 \tag{3.90}$$

In order to find the mean effects the time average of (3.86) to (3.90) is taken to find that:

$$\epsilon_o \nabla\cdot\overline{\vec{u}\, u} = -\overline{p}_x \tag{3.91}$$

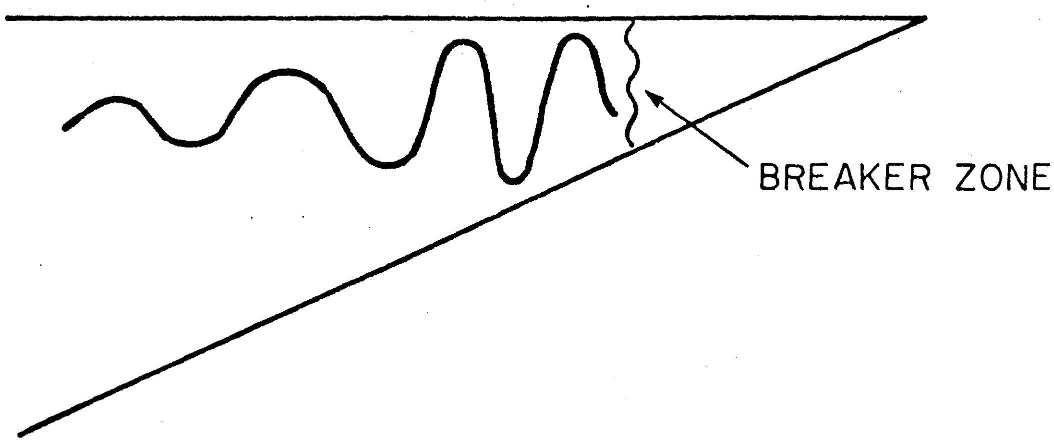
$$\epsilon_o \nabla\cdot\overline{\vec{u}\, v} = -\overline{p}_y \tag{3.92}$$

$$\epsilon_o \nabla\cdot\overline{\vec{u}\, w} = -\overline{p}_z - \overline{\rho} \tag{3.93}$$

$$\epsilon_o \nabla\cdot\overline{\vec{u}\, \rho} = \overline{w} \tag{3.94}$$

$$\nabla\cdot\overline{\vec{u}} = 0 \tag{3.95}$$

The quadratic quantities in this set are Reynolds stresses which are produced by the fundamental wave motion and act on the mean properties. Garrett (1968) has been able to relate these correlations to a generalized radiation stress for plane waves propagating through a shear flow in an infinite fluid. The linear flat bottom modal solutions are such that the divergence of these stresses vanishes in the averaged horizontal momentum equations making the horizontal derivatives of the mean



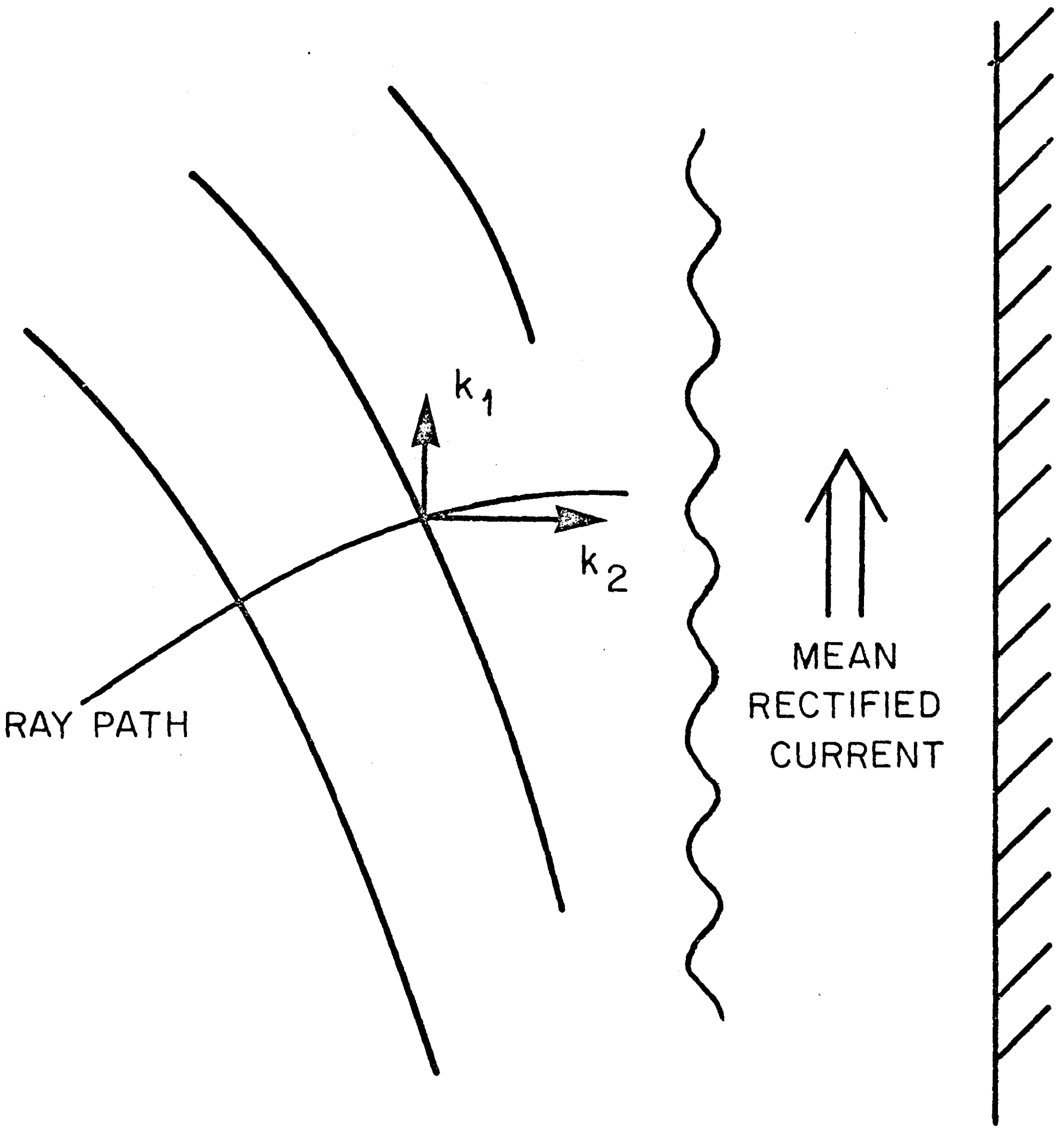




Figure 25. Longshore current generation.

pressure vanish to $O(\epsilon_o)$. However, if the linear wave is made to vary on some scale L that is large compared with a wavelength $\lambda$ then the horizontal divergence of these quadratic stresses is $O(\lambda/L)$ and not zero. The remaining part of the divergence is a z-derivative of a correlation that vanishes to O(1) because the horizontal and vertical velocity components are out of phase for the linear flat bottom solutions. This stress component, and therefore the individual velocity components, must be known to $O(\lambda/L)$ in order to determine the nonlinear effects of the wave field on the mean properties of the medium.

Bretherton (1969), in his analysis of plane internal waves in an infinite atmosphere chooses the scale L to be the scale of a wave packet. In this analysis the waves are made to vary slowly in their spacial dimensions by having a steady train of waves shoal on a beach which changes depth significantly on a scale L which is much greater than a wavelength. The $O(\lambda/L)$ influence of this beach is found by a WKB expansion similar to that used in section (3.3).

The linear problem determining the $O(\lambda/L)$ influence of the bottom slope on the waves is a slightly simplified version of the shoaling internal wave problem solved by Keller and Mow (1969). Their problem included surface movements, an arbitrary beach geometry, and an exponential density profile in place of the linear one assumed here.

After setting $\epsilon_o = 0$ in equations (3.86) to (3.90), time can be eliminated through the substitution:

$$\vec{u}(\vec{x},t) = Re\left\{\vec{U}(\vec{x})\, e^{-i\omega t}\right\}$$

$$p(\vec{x},t) = Re\{P(\vec{x})e^{-i\omega t}\}$$
$$\rho(\vec{x},t) = Re\{R(\vec{x})e^{-i\omega t}\} \tag{3.96}$$

so that:

$$-i\omega U = -P_x \tag{3.97}$$
$$-i\omega V = -P_y \tag{3.98}$$
$$-i\omega W = -P_z - R \tag{3.99}$$
$$-i\omega R = W \tag{3.100}$$
$$U_x + V_y + W_z = 0 \tag{3.101}$$

From (3.97) to (3.100) U, V, W and R can be written in terms of P. Accordingly:

$$U = -\frac{i}{\omega} P_x \tag{3.102}$$
$$V = -\frac{i}{\omega} P_y \tag{3.103}$$
$$W = \frac{i c^2}{\omega} P_z \tag{3.104}$$
$$R = -\frac{c^2}{\omega^2} P_z \tag{3.105}$$

and substitution into (3.101) yields a single equation in pressure.

$$\nabla_h^2 P - c^2 P_{zz} = 0 \tag{3.106}$$

The top and bottom boundary conditions (3.16) can also be rewritten in terms of P by use of (3.102) and (3.104). In this way:

$$P_z = 0 \quad \text{at} \quad z = 0 \tag{3.107}$$
$$P_z = \frac{\alpha}{c^2} P_x h_X \quad \text{at} \quad z = -h(X) \tag{3.108}$$

where the bottom slope is charactarized by a small parameter and a stretched x-coordinate $X=\alpha x$. Although it is not necessary the beach has been taken to depend only on the x-coordinate. Keller and Mow (1969) deal with the more general beach configuration.

The idea now is to expand $P(\vec{x})$ in a WKB expansion of the form:

$$P(\vec{x}) = \sum_{n=0}^{\infty} \alpha^n P^{(n)}(X,z) e^{i\psi(x,y)} \tag{3.109}$$

where

$$\begin{aligned} \vec{k}(X,y) &= (k_1, k_2) \\ &= \nabla_h \psi . \end{aligned} \tag{3.110}$$

The wavenumber is also expanded in a power series in $\lambda$. Because both $P^{(n)}$ and $\vec{k}$ are treated as complex quantities they carry redundant information part of which is removed by using only the even powers for $\vec{k}$. That is:

$$\vec{k}(X,y) = \sum_{n=0}^{\infty} \alpha^{2n} \vec{k}^{(2n)}(X,y) \tag{3.111}$$

Substitution of these power series expansions into equation (3.106) and the boundary conditions (3.107) and (3.108) yields a sequence of problems which are summarized below to $O(\alpha)$.

<u>$O(\alpha^0)$:</u> This is the usual flat bottom internal wave problem. The pressure equation is:

$$-\left(k_1^{2(0)} + k_2^{2(0)}\right) P^{(0)} - c^2 P_{zz}^{(0)} = 0 \tag{3.112}$$

with conditions: $\quad P_z = 0 \quad \text{at} \quad z = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. (3.113)

This describes an eigenvalue problem which possesses solutions for specific values of $k_1^{2(0)}+k_2^{2(0)}$. A typical mode is:

$$P^{(0)}(X,z) = A^{(0)}(X)\cos\frac{\ell\pi z}{h(X)} \tag{3.114}$$

with
$$k_1^{2(0)} + k_2^{2(0)} = \left(\frac{c\,\ell\pi}{h(X)}\right)^2 \tag{3.115}$$

and $\ell$ an integer.

An additional constraint is necessary on the wavenumber relation in order to determine $k_1^{(0)}$ and $k_2^{(0)}$ separately. The required expression is:

$$\begin{aligned}\nabla\times\vec{k} &= \hat{k}\,(k_{2x} - k_{1y}) \\ &= 0\end{aligned} \tag{3.116}$$

which demands that wave crests be conserved. This implies that $k_{2x}^{(0)}=0$ as $k_1^{(0)}$ has no y dependence. Therefore the longshore wavenumber component is constant along rays as the wave progresses up the slope and the onshore component increases according to the relation:

$$k_1^{(0)} = \left\{\left(\frac{c\,\ell\pi}{h(X)}\right)^2 - k_2^{2(0)}\right\}^{1/2} \tag{3.117}$$

As the wavelength in the x-direction decreases while the wave shoals and that in the y-direction remains constant the wave is refracted so that crests tend to line up parallel to the beach in a manner that is analogous to surface waves.

<u>$O(\alpha)$:</u> At this order the first effects of the slowly changing bottom on the wave amplitude are seen. Equation (3.106) to this

level of approximation is:

$$P_{zz}^{(1)} + \frac{|\vec{k}^{(0)}|^2}{c^2} P^{(1)} = \frac{i}{c^2}\left( k_{1X}^{(0)} P^{(0)} + 2\, k_1^{(0)} P_X^{(0)} \right)$$
$$= \frac{i}{c^2}\Big[ \left( k_{1X}^{(0)} A^{(0)} + 2 k_1^{(0)} A_X^{(0)} \right) \cos \frac{\ell \pi z}{h} + {} + 2 k_1^{(0)} h_X A^{(0)} \frac{\ell \pi z}{h^2} \sin \frac{\ell \pi z}{h} \Big] \qquad (3.118)$$

and the boundary conditions are:

$$P_z^{(1)} = 0 \quad \text{at } z = 0, \qquad (3.119)$$
$$P_z^{(1)} = \frac{i}{c^2} P^{(0)} k_1^{(0)} h_X \quad \text{at } z=-h(X). \qquad (3.120)$$

The solution to (3.118) is of the form:

$$P^{(1)}(X,z) = b_1(X)\, z^2 \cos \frac{\ell \pi z}{h} + b_2(X)\, z \sin \frac{\ell \pi z}{h} \qquad (3.121)$$

where $b_1(X)$ and $b_2(X)$ can be determined by substitution into the governing equation (3.118). One finds that:

$$b_1(X) = -\frac{1}{2} \frac{i}{c^2} \frac{k_1^{(0)} h_X}{h} A^{(0)}, \qquad (3.122)$$
$$b_2(X) = \frac{h}{2m\pi}\left(\frac{i}{c^2}\right)\left( k_1^{(0)} \frac{h_X}{h} A^{(0)} + 2 k_1^{(0)} A_X^{(0)} + k_{1X}^{(0)} A^{(0)} \right). \qquad (3.123)$$

There is a solution to the homogeneous equation with homogeneous boundary conditions that has been omitted for the same reasons that $\vec{k}$ was expanded in an even power series.

$P^{(1)}(X,z)$ must also satisfy the boundary conditions of (3.119) implying that:

$$b_2(X) = 0$$

and

$$k_{1X}^{(0)} A^{(0)} + 2 k_1^{(0)} A_X^{(0)} + k_1^{(0)} \frac{h_X}{h} A^{(0)} = 0 .$$

On substituting equation (3.117) for $k_1^{(0)}$ into this relation the amplitude function is determined to be:

$$A_X^{(0)} = 0$$

or

$$A^{(0)} = \text{constant} \tag{3.124}$$

The pressure is now known to $O(\alpha)$ allowing for the calculation of the other variables to this order from equations (3.102) to (3.105). Therefore:

$$p(\vec{x},t) = A^{(0)} \cos \frac{\ell \pi z}{h} \cos(\psi - \omega t) + \\ + \alpha \frac{A^{(0)}}{c^2} \frac{k_1^{(0)} h_X}{2h} z^2 \cos \frac{\ell\pi z}{h} \sin(\psi - \omega t) + O(\alpha^2) \tag{3.125}$$

$$u(\vec{x},t) = \frac{k_1^{(0)} A^{(0)}}{c^2} \cos \frac{\ell\pi z}{h} \cos(\psi - \omega t) + \\ + \alpha \frac{k_1^{(0)} A^{(0)}}{\omega c^2} \left\{ \frac{k_1^{(0)} h_X}{2h} z^2 \cos \frac{\ell\pi z}{h} - k_{1X}^{(0)} \frac{hz}{2\ell\pi} \sin \frac{\ell\pi z}{h} \right\} \sin(\psi - \omega t) + \\ + O(\alpha^2) \tag{3.126}$$

$$v(\vec{x},t) = \frac{k_2^{(0)} A^{(0)}}{\omega} \cos \frac{\ell\pi z}{h} \cos(\psi - \omega t) + \\ + \alpha \frac{k_2^{(0)} A^{(0)}}{\omega c^2} \left\{ \frac{k_1^{(0)} h_X}{2h} z^2 \cos \frac{\ell\pi z}{h} \right\} \sin(\psi - \omega t) + O(\alpha^2) \tag{3.127}$$

$$w(\vec{x},t) = \frac{c^2 A^{(0)}}{\omega} \frac{\ell\pi}{h} \sin \frac{\ell\pi z}{h} \sin(\psi - \omega t) + \\ + \frac{\alpha A^{(0)}}{\omega} \left\{ - \frac{k_1^{(0)} h_X}{2h} \frac{\ell\pi z^2}{h} \sin \frac{\ell\pi z}{h} + \frac{k_1^{(0)} h_X}{h} z \cos \frac{\ell\pi z}{h} \right\} \cos(\psi - \omega t) + \\ + O(\alpha^2) \tag{3.128}$$

$$\rho(\vec{x},t) = \frac{c^2 A^{(0)}}{\omega^2} \cdot \frac{\ell\pi}{h} \sin \frac{\ell\pi z}{h} \cos(\psi - \omega t) - \\ - \alpha \frac{A^{(0)}}{\omega} \left\{ - \frac{k_1^{(0)} h_X}{2h} \cdot \frac{\ell\pi z^2}{h} \sin \frac{\ell\pi z}{h} + \frac{k_1^{(0)} h_X}{h} z \cos \frac{\ell\pi z}{h} \right\} \cos(\psi - \omega t) + \\ + O(\alpha^2) \tag{3.129}$$

These expressions can now be used to calculate the Reynolds stresses and buoyancy flux terms to $O(\alpha)$. These are:

$$\overline{uu} = \frac{1}{2}\left(\frac{k_1^{(0)}A^{(0)}}{\omega}\cos\frac{\ell\pi z}{h}\right)^2 + O(\alpha^2) \tag{3.130}$$

$$\overline{uv} = \frac{1}{2}k_1^{(0)}k_2^{(0)}\left(\frac{A^{(0)}}{\omega}\cos\frac{\ell\pi z}{h}\right)^2 + O(\alpha^2) \tag{3.131}$$

$$\overline{uw} = \frac{1}{2}\alpha\frac{A^{2(0)}}{\omega^2}\frac{h_X}{h}k_1^{2(0)}z + O(\alpha^2) \tag{3.132}$$

$$\overline{vv} = \frac{1}{2}\left(\frac{k_2^{(0)}A^{(0)}}{\omega}\cos\frac{\ell\pi z}{h}\right)^2 + O(\alpha^2) \tag{3.133}$$

$$\overline{vw} = \frac{1}{2}\alpha k_1^{(0)}k_2^{(0)}\frac{h_X}{h}z\left(\frac{A^{(0)}}{\omega}\cos\frac{\ell\pi z}{h}\right)^2 + O(\alpha^2) \tag{3.134}$$

$$\overline{ww} = \frac{1}{2}\left(\frac{c^2A^{(0)}}{\omega}\frac{\ell\pi}{h}\sin\frac{\ell\pi z}{h}\right)^2 + O(\alpha^2) \tag{3.135}$$

$$\overline{u\rho} = \frac{1}{4}k_1^{(0)}\left(\frac{cA^{(0)}}{\omega}\right)^2\frac{\ell\pi}{\omega h}\sin\frac{\ell\pi z}{h} + O(\alpha^2) \tag{3.136}$$

$$\overline{v\rho} = \frac{1}{4}k_2^{(0)}\left(\frac{cA^{(0)}}{\omega}\right)^2\frac{\ell\pi}{\omega h}\sin\frac{2\ell\pi z}{h} + O(\alpha^2) \tag{3.137}$$

$$\overline{w\rho} = O(\alpha^2) \tag{3.138}$$

Note that the stress components $\overline{uw}$ and $\overline{vw}$ are $O(\alpha)$ so that the z-derivatives are $O(\alpha)$ and of the same order as the horizontal derivatives.

The Reynolds stresses are now determined to $O(\alpha)$ and the averaged nonlinear equations (3.91) to (3.95) can be returned to for a calculation of the mean pressure, density, and velocity. Equations (3.94) and (3.95) form a closed system from which $\bar{u}$ and $\bar{w}$ can be found to be:

$$\begin{aligned}\bar{w} &= \epsilon_0\nabla\cdot\overline{\vec{u}\rho} \\ &= \epsilon_0\alpha\frac{\ell\pi}{4\omega}\left(\frac{cA^{(0)}}{\omega}\right)^2\frac{\partial}{\partial X}\left(\frac{k^{(0)}}{h}\sin\frac{2\ell\pi z}{h}\right)\end{aligned} \tag{3.139}$$

and

$$\bar{u} = -\frac{\epsilon_0}{2\omega}\left(\frac{cA^{(0)}}{\omega}\right)^2 k^{(0)}\cdot\left(\frac{\ell\pi}{h}\right)^2\cos\frac{2\ell\pi z}{h}\,. \tag{3.140}$$

The equation for $\bar{u}$ is a generalization of equation (3.45) for the mean Eulerian velocity in normally incident waves. The amplitude factor here $A^{(0)}=2\omega A^{(1,1)}$ of section (3.3).

Equation (3.92) is trivial as $\bar{p}_y=0$ from the uniform beach assumption and it can easily be shown that $\nabla\cdot\overline{\vec{u}v}=0$. The pressure can be eliminated from (3.91) and (3.93) by cross differentiation:

$$\bar{\rho}_X = \epsilon_0\left(\frac{\partial}{\partial z}\nabla\cdot\overline{\vec{u}u} - \frac{\partial}{\partial X}\nabla\cdot\overline{\vec{u}w}\right)$$

$$= \epsilon_0\left\{\left(\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial X^2}\right)\overline{uw} + \frac{\partial^2}{\partial X\partial z}\left(\overline{u^2} - \overline{w^2}\right)\right\} \qquad (3.141)$$

Because $\overline{uw}$ is $O(\alpha)$ and a linear function of z the term containing it in (3.141) is of a lower order than the others and contributes nothing. The labor of the WKB analysis involved in evaluating the $\overline{uw}$ correlation to $O(\alpha)$ has therefore only shown that it does not affect the mean isopycnal shift $\bar{\rho}$.

Integrating in x, (3.141) gives:

$$\bar{\rho}(X,z) = -\frac{1}{2}\epsilon_0\left(\frac{A^{(0)}}{\omega}\right)^2\frac{\ell\pi}{h}\left(k_1^{2(0)} + c^2\left|\vec{k}^{(0)}\right|^2\right)\sin\frac{2\ell\pi z}{h} \qquad (3.142)$$

which reduces to (3.67) when $k_2^{(0)}=0$. An arbitrary function of z arising from the integration has been chosen to vanish for the same reasons used in the normal incidence case.

Now, if the wavenumber vector at some point on the slope makes an angle $\theta$ with the x-axis then $k_1^{(0)}$ and $k_2^{(0)}$ can be written in the form:

$$k_1^{(0)} = \left|\vec{k}^{(0)}\right|\cos\theta\,,$$
$$k_2^{(0)} = \left|\vec{k}^{(0)}\right|\sin\theta\,. \qquad (3.143)$$

Letting the initial flat bottom angle be $\theta_0$ then:

$$\frac{\sin\theta}{h(X)} = \frac{\sin\theta_o}{h_o} \tag{3.144}$$

from the fact that $k_2^{(0)}$ does not change. This is a Snell's law for internal gravity waves showing that as h(X) decreases $\sin\theta$ and $\theta$ must also decrease and the wave crests tend to become parallel to the beach contours.

Equation (3.144) can now be used to show that:

$$k_1^{(0)} = |\vec{k}^{(0)}| \left(1 - \left(\frac{h(X)\sin\theta_o}{h_o}\right)^2\right)^{1/2} \tag{3.145}$$

and (3.142) for $\bar{\rho}(X,z)$ can be rewritten as:

$$\bar{\rho}(X,z) = -\frac{1}{2}\left(\frac{A^{(0)}}{\omega}\right)^2 \frac{|\vec{k}^{(0)}|^3}{c}\left[\frac{1}{\omega^2} - \left(\frac{h(X)\sin\theta_o}{h_o c}\right)^2\right]\sin\frac{2\ell\pi z}{h(X)} \tag{3.146}$$

The mean density shift is reduced by a factor of $1 - \left(\frac{\omega}{c}\frac{h}{h_o}\sin\theta_o\right)^2$ over that obtained in the normal incidence problem. This factor approaches unity as the depth h(X) decreases allowing one to conclude that oblique incidence will not have a radical effect on the magnitude of the isopycnal displacements.

There is, however, one interesting feature of this problem. The picture up to this point has been that the waves gradually refract as they progress up the beach, the wavelength shortens and the amplitude of the motion increases. This steepening of the waves can only continue to a certain point where the waves become unstable and break. Cacchione (1970) reported on experiments with normally incident high frequency internal waves in which this breaking occurred internally in a manner analogous to the breaking of surface waves. In particular the breaking appeared to take place within a well defined breaker zone.

Between this breaker zone and the corner the motion is governed by dissipation as well as nonlinear forces and it is possible for the oscillatory motion of the wave to be rectified into a steady longshore current that is limitted in magnitude only by the dissipation. The situation in mind is represented pictorially in figure (25). Bowen (1969) and Longuet-Higgins(1970) analyse the corresponding surface wave problem and their methods are followed closely in what follows.

When dissipative forces , produced by viscosity and heat conduction are included in the time averaged equations (3.91) to (3.95) one has the set:

$$\epsilon_0 \nabla\cdot\overline{\vec{u}u} = -\bar{p}_x + F_1(\bar{u}) \tag{3.147}$$

$$\epsilon_0 \nabla\cdot\overline{\vec{u}v} = F_2(\bar{v}) \tag{3.148}$$

$$\epsilon_0 \nabla\cdot\overline{\vec{u}w} = -\bar{p}_z - \bar{\rho} + F_3(\bar{w}) \tag{3.149}$$

$$-\bar{w} + \epsilon_0 \nabla\cdot\overline{\vec{u}\rho} = R(\bar{\rho}) \tag{3.150}$$

$$\bar{u}_x + \bar{w}_z = 0 \tag{3.151}$$

The assumption of no y variations of mean quantities is retained.

Outside the breaker zone in the region where the previous results are assumed to hold $\nabla\cdot\overline{\vec{u}v} = 0$ and therefore $F_2(\bar{v})=0$. If a simple eddy viscosity model is chosen for the dissipation with:

$$F_2(\bar{v}) = E\,\frac{\partial^2 \bar{v}}{\partial z^2} \tag{3.152}$$

in particular, then $\bar{v} = 0$ before breaking. Inside the breaker zone, however, the expressions derived for the correlations are no longer valid and $\nabla\cdot\overline{\hat{u}v}$ is not necessarily zero and a mean current can be driven by this Reynolds stress which is balanced by the viscous dissipation.

The idea used in the surface wave problem is to retain the form of the correlations and to make an ad hoc assumption about the amplitude of the motion. Away from the dissipation area conservation of energy determines the local amplitude but, after breaking, the motion must decay to zero in the corner. A simple form is:

$$a(X) = \gamma\, h(X) \tag{3.153}$$

where $a(X)$ is the wave amplitude and $\gamma$ a constant of proportionality.

This amplitude is related to the pressure amplitude $A^{(0)}(X)$ by:

$$a(X) = \frac{c^2 A^{(0)}\, \ell\pi}{\omega^2 \quad h(X)} . \tag{3.154}$$

showing that the amplitude grows as $h^{-1}(X)$ before breaking. In terms of $A^{(0)}$ the assumption in (3.153) is:

$$A^{(0)}(X) = \gamma \frac{\omega^2}{c^2} \frac{h^2(X)}{\ell\pi} . \tag{3.155}$$

When the wave breaks the crests will be almost parallel to the beach so that $k_1^{(0)} \sim |\vec{k}^{(0)}|$. Substituting (3.155) into (3.131) and (3.134) and then the result into (3.152) yields:

$$\bar{v} = \left(\frac{\alpha \epsilon_o \gamma^2 \omega^2}{2 E c^2}\right)\left(\frac{k_2^{(0)}}{k_1^{(0)}} h^3 |h_x|\right)\left(1 - \left(\frac{z}{h}\right)^2 - \frac{\sin^2 \frac{\ell \pi z}{h}}{\ell^2 \pi^2}\right) \tag{3.156}$$

after the double integration in the vertical. The two constants of integration were defined by a stress free condition on the surface and by having the velocity vanish on the bottom.

It is still necessary to know the position of the breaker line before $\gamma$ and $\bar{v}$ can be estimated. One possible mechanism for breaking is the development of a shear instability. The O(1) Richardson number, Ri, as determined from equation (3.126) is:

$$Ri = \left(\frac{c^2 N}{a \omega k_1^{(0)}}\right)^2$$

Setting this equal to 1/4 one finds that:

$$\epsilon = k_1^{(0)} a = 2 \frac{c^2 N}{\omega}$$

or the wave steepness parameter is approximately unity when the Richardson number indicates instability. Using this as the criterion for breaking (i.e. $\epsilon = 1$) , the fluid depth at the breaker zone is:

$$h_b = h_o (a_o |\vec{k}_o^{(0)}|)^{1/2}. \tag{3.157}$$

For an initial wave amplitude of 10m and wavenumber magnitude of $h_o^{-1}$ in a flat bottom depth of 2km, $h_b \approx 140$m. At this point the O(1) wave amplitude is equal to the depth or $a_b \approx 140$m meaning that the constant $\gamma = 1$ from (3.153).

Equation (3.156) for $\bar{v}$ in dimensional form is:

$$\bar{V} = \frac{\gamma^2\omega^2}{2Ec^2} \cdot \frac{k_2^{(0)}}{k_1^{(0)}} h^3 |h_x| \cdot \left(1 - \left(\frac{z}{h}\right)^2 - \frac{\sin\frac{\ell\pi z}{h}}{\ell^2\pi^2}\right) . \tag{3.158}$$

Now choosing $\omega = N = 10^{-3}\mathrm{sec}^{-1}$, $c = 1$, $h_x = .2$, and

$$\frac{k_2^{(0)}}{k_1^{(0)}} = \frac{\frac{h}{h_0}\sin\theta_0}{\left(1 - \left(\frac{h}{h_0}\sin\theta_0\right)^2\right)^{1/2}} \sim \frac{h_b}{h_0} = .07 \tag{3.159}$$

from (3.145), one finds that:

$$\bar{V} \cong \frac{10^5}{E}\left(1 - \left(\frac{z}{h}\right)^2 - \frac{\sin\frac{\ell\pi z}{h}}{\ell^2\pi^2}\right) \mathrm{cm/sec}. \tag{3.160}$$

For any reasonable values of the eddy viscosity this could be a sizeable flow. A large number of rather crude approximations have gone into the final result so that it ought to be treated as an order of magnitude estimate only but it does indicate that measureable currents can be produced along the beach by refracting internal waves. Bowen (1969) and Bowen and Inman (1969) have also studied rip currents forced by the interaction of normally incident surface waves with edge waves and achieved good comparison between theory and observation. Such results could carry over to the internal wave regime but the lack of any observations, either field or experiment, reduces the value of such an extension at this time.

# 4. CONCLUSIONS

The steady flow of an inviscid, stratified ocean on an f-plane around an island of small aspect ratio that is modeled by a cylinder with steep sides gives isotherm displacements which agree well with those observed on two cruises to the Bermuda area. To $O(\epsilon^0)$ potential vorticity is conserved and the flow responds to a thermal wind balance as it diverges past the island. For a current with shear in one direction but which reverses direction at some depth the thermal wind adjustment leads to an area of depressed isotherms on the left extreme of the island, looking downstream, and an elevated area to the right. This pattern was observed on cruise Atlantis 11 47.

During the later part of Gosnold 144 the current was stronger and to the east at all depths making first order Rossby number effects comparable in magnitude to those produced by the slope. The changes in potential vorticity now give a depressed area near the forward stagnation point and a hill to the south in agreement with the data. The superposition of the small slope and Rossby number solutions also lead to an small observed elevation in the 17°C isotherm on the northern extreme of the island.

Simple Richardson number stability arguments show that the flow, on the large scale, is stable everywhere, but that Ri reaches a minimum (about 1/4 the upstream value) on the north slope for an easterly flow. If there are local areas in the water column with a low enough stability then they could be pushed to the critical point in this region. In addition, an investigation of the boundary layer shows that, if streamline separation from the boundary does occur, it is most likely to happen on the north slope. It is in this region that the greatest amount of microstructure is observed.

There are a number of complications to the flow at Bermuda such as nonuniformity, unsteadiness, and a complicated geometry which reduce the agreement between fact and theory. To test the ideas presented here there exist a number of other islands which more closely approach the ideal. Ascension I. in the South Atlantic is nearly conical and in the steady, uniform, South Equatorial Current. Some of the Maldive Islands in the Indian Ocean and island such as Jarvis and Baker on the Equatorial Undercurrent of the Pacific are also suitable. The last two, however are rather close to the equator and some adjustments of the theory would be necessary to account for the vanishing of the vertical component of the earth's rotation.

Internal gravity waves, shoaling on a plane beach, can also produce mean isotherm shifts but these displacements have a modal form and the mean isotherm is shifted both up and down at any horizontal position on the beach. This effect is observed in the initial current regime of Gosnold 144 but, because of lack of data, the calculated results are not significant with respect to the theoretically large amplitude of the waves that would have to be present. Within the scope of these ideas the extensive microstructure is taken to be evidence of breaking.

The generalized WKB analysis of the normally incident waves gives additional information which could be verified in future laboratory experiments of the problem. Such quantities as the wavenumber correction to second order in the wave slope and bottom slope and second harmonic generation are calculated.

Study of the mean effects of obliquely incident waves shows that the amplitude of the mean isotherm displacements is not much affected by this generalization. From these results it is shown that a significant longshore current can be forced behind the breaker zone.

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